Lecture notes

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Preface

These lecture notes are a compilation of problems, ideas, useful facts, and theorems that cover the basics for preparation for math competitions. They are not meant to be exhaustive. They are not meant to provide a sound mathematical foundation in mathematics. In fact, the text relies on certain familiarity with the main topics discussed. It is meant as a collection of essentials needed for the IMO and other high school mathematics competitions.

The selected problems are either instructive—demonstrating a new concept—or interesting in their own right. This is the reason why sometimes disproportionately hard problems are mixed with disproportionately easy ones, which is still in line with the objective of the exercise: to teach new concepts and to promote curiosity in the subject.

Since the IMO problems are usually categorized into one of the four areas: Algebra, Geometry, Combinatorics, and Number Theory, we have adopted the same nomenclature here.

In their current form, the lecture notes are far from complete. Suggestions for problems, topics, or improvements are very welcome.

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Chapter 1

Algebra

The topic of algebra roughly deals with algebra typically involves manipulation of real numbers, polynomials, functional equations, and inequalities.

1.1 Inequalities

Inequalities appear frequently in mathematical competitions and require a combination of technique and insight. Learning to solve them is a matter of practice, but there are some general hints.

Classical Classical inequalities such as AM–GM and Cauchy–Schwarz often provide a key step in solving many problems, though they are rarely the only thing we need. These inequalities should be thoroughly familiar and readily applicable.

Substitutions It might help to change the variables in question from x, y, z to

$$(x, y, z) = \left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right),$$
$$(x, y, z) = (a + b, b + c, a + c),$$
$$(x, y, z) = (\tan(\alpha), \tan(\beta), \tan(\gamma)),$$
$$(x, y, z) = (b + c - a, a + c - b, a + b - c),$$
$$(x, y, z) = \left(\frac{b + c}{2}, \frac{c + a}{2}, \frac{a + b}{2}\right)$$

Convexity Use Jensen's inequality for convex functions.

Smoothing This is a great method that usually gets too little attention, so we shall look at it in more detail.

Varying Use the derivative of the function to change x_i in the direction of maximum or minimum. Once there, examine the equality. A cleaner alternative to varying are LaGrange's multipliers.

1.1.1 Classical

These inequalities are must-learn and one should be very familiar with their scope of application as well as with their limitations. In a sence, they are week inequalities (but with wider scope for that matter) and should not be attempted as a starting step. Rather, they should be used at some intermediate step in the solition.

Theorem 1.1.1. *(Cauchy-Schwarz) For any real* $a_1, a_2, \ldots, a_n$ *and* $b_1, b_2, \ldots, b_n$*, the following inequality holds*

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \geq \left(\sum_{i=1}^{n} a_i b_i \right)^2$$

with equality holding iff the vectors $(a_1, a_2, \ldots, a_n)$ *and* $(b_1, b_2, \ldots, b_n)$ *are linearly dependent.*

Theorem 1.1.2. *(AM-GM) For any non-negative real numbers* $x_1, x_2, \ldots, x_n$ *holds*

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n}.$$

Theorem 1.1.3. *(Maclaurin) Let* $x_1, x_2, \ldots, x_n$ *be non-negative real numbers. For each* $k = 1, 2, \ldots, n$*, denote*

$$c_k = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$$

and let $d_k = \frac{c_k}{\binom{n}{k}}$*. Prove that*

$$d_1 \geq \sqrt[2]{d_2} \geq \sqrt[3]{d_3} \geq \cdots \geq \sqrt[n]{d_n}.$$

Theorem 1.1.4. *(Schur) For any* $r > 0$ *and non-negative* $x, y,$ *and* z *holds*

$$x^r(x-y)(x-z) + y^r(y-z)(y-x) + z^r(z-x)(z-y) \geq 0.$$

Theorem 1.1.5. *(Chebyshev) If* $x_1 \geq x_2 \geq \cdots \geq x_n$ *and* $y_1 \leq y_2 \leq \cdots \leq y_n$ *are all real numbers, then*

$$\frac{\sum_i x_i y_{n+1-i}}{n} \geq \frac{\sum_i x_i}{n} \frac{\sum_i y_i}{n} \geq \frac{\sum_i x_i y_i}{n}.$$

Theorem 1.1.6. *(Hoelder) For any positive* p *and* q *with* $p + q = pq$ *and non-negative* x_i *and* y_i *holds*

$$\sum_{i=1}^{n} x_i y_i \leq \left(\sum x_i^p \right)^{\frac{1}{p}} \left(\sum y_i^q \right)^{\frac{1}{q}}.$$

Proposition 1.1.1.

$$\sum_{i=1}^{n} \sqrt{x_i^2 + y_i^2 + \cdots + z_i^2} \geq \sqrt{\left(\sum_{i=1}^{n} x_i \right)^2 + \left(\sum_{i=1}^{n} y_i \right)^2 + \cdots + \left(\sum_{i=1}^{n} z_i \right)^2}.$$

1.1.2 Convexity

If a function is convex, then the line connecting any two points on it lies "above" the function. This is a simple geometrical observation leading to a simple but extremely powerful algebraical statement.

A test to check whether a differentiable function f is convex is $f''(x) \geq 0$. If that is the case, we can apply the following property, known as Jensen's inequality:

Proposition 1.1.2. (Jensen's inequality) If f is a convex function, then for any x_1 and x_2 from the domain of f and for any $\lambda \in (0, 1)$ holds

$$\lambda f(x_1) + (1 - \lambda)f(x_2) \geq f(\lambda x_1 + (1 - \lambda)x_2).$$

1.1.3 Smoothing

Examining when the equality case holds, we can pretty much guess whether it happens when all the variables are equal (symmetric case) or when they are as far apart as it gets (asymmetric case.) The question is, is it possible to start with any numbers and start moving them closer together (or further apart) so that the inequality becomes tighter. If we can, we are in luck, since we need only examine the extreme case when all variables are equal (or when they are at their lowest or highest limits.) The following example illustrates this interesting concept:

Proposition 1.1.3. For any real $a_1, a_2, \ldots, a_n$ is valid

$$n(a_1^2 + a_2^2 + \cdots + a_n^2) \geq (a_1 + a_2 + \cdots + a_n)^2$$

with equality holding when $a_1 = a_2 = \cdots = a_n$ (in other words, the sum of squares is minimized for a fixed sum when the numbers are equal.)

1.1.4 Varying and LaGrange's multipliers

Suppose we are given a general inequality of the form $f(x_1, x_2, \ldots, x_n) \geq 0$. All inequalities look like this anyway, since we can always subtract the rigt side from the left side. If we want to prove the inequality for all points $X = (x_1, x_2, \ldots, x_n)$ belonging to some space A, it is probably worth proving it for the point where f reaches its minimum on A. If it holds there, it holds everywhere on A.

The question becomes how to find the minimum of f on A. Notice, that if we are allowed to vary x_i by "a little bit" while staying in A, the function f can only increase locally, so the point of the minium must be a local minimum. If, on the other hand, x_i cannot be varied freely, that must be because x_i is on the "surface" of A. In either case, we only need to consider local minimums or points on the surface of A, which already simplifies the problem.

What we know for every local minimum is that the partial derivatives of f are zero there, so we can seek to solve the system of equations

$$\left|\frac{\partial f}{\partial x_i} = 0\ , \qquad i = 1, 2, \ldots, n.\right.$$

When there is an additional constraint, for instance $x_1 + x_2 + \cdots + x_n = 1$, the question becomes a bit harder but not fundamentally different. We can write the constraint as $g(x_1, x_2, \ldots, x_n) = 0$ and the general problem becomes: minimize f over A under $g = 0$. What LaGrange says, and we shall believe him, is that all minimal solutions $x_1, x_2, \ldots, x_n$ which are not on the border of A satisfy

$$\left| \frac{\partial f}{\partial x_i} = \lambda \,, \qquad i = 1, 2, \ldots, n\right.$$

for some real λ. With the additional condition $g(x_1, x_2, \ldots, x_n) = 0$, we have a system with $n+1$ equation and $n+1$ unknowns (including λ). Once we solve the system and examine the solutions, which gives us the local minima of f, we need to see what happens if one of x_i is on the surface.

Consider the following simple example

Proposition 1.1.4. The maximum and minimum of $a\cos(\phi) + b\sin(\phi)$ are $\pm\sqrt{a^2+b^2}$ and are achieved when $\cos(\phi) = \pm\dfrac{a}{\sqrt{a^2+b^2}}$.

1.1.5 Exercises

Problem 1.1.1. Prove that for any three positive numbers $a, b,$ and c holds

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{3}{2}.$$

Problem 1.1.2. (IMO, 1983) Let $a, b,$ and c be the lengths of the sides of a triangle. Prove that

$$a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \geq 0.$$

Problem 1.1.3. (Kvant, 1985) Prove that for any four positive numbers $a, b, c,$ and d holds

$$\frac{a}{b+c} + \frac{b}{c+d} + \frac{c}{d+a} + \frac{d}{a+b} \geq 2.$$

Problem 1.1.4. (IMO, 1995) Let a, b, c be positive real numbers with $abc = 1$. Prove that

$$\frac{1}{a^3\,(b+c)} + \frac{1}{b^3\,(c+a)} + \frac{1}{c^3\,(a+b)} \geq \frac{3}{2}.$$

Problem 1.1.5. (Singapore MO, 2008) Prove that for all real non-negative a, b, c holds

$$\frac{(1+a^2)(1+b^2)(1+c^2)}{(1+a)(1+b)(1+c)} \geq \frac{1+abc}{2}.$$

Problem 1.1.6. (IMO, 1999) Let a, b, c be positive real numbers so that $abc = 1$. Prove that

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1.$$

Problem 1.1.7. (IMO, 2001) Prove that for all positive real numbers a, b, c,

$$\frac{a}{\sqrt{a^2+8bc}} + \frac{b}{\sqrt{b^2+8ca}} + \frac{c}{\sqrt{c^2+8ab}} \geq 1.$$

Problem 1.1.8. (IMO, 2004) Let $r_1, r_2, \ldots, r_n$ be real numbers greater than or equal to 1. Prove that

$$\frac{1}{r_1+1} + \frac{1}{r_2+1} + \cdots + \frac{1}{r_n+1} \geq \frac{n}{\sqrt[n]{r_1 r_2 \cdots r_n}+1}.$$

Problem 1.1.9. (IMO, 1999) Let $n \geq 2$ be a fixed integer. Find the least constant C such that the inequality

$$\sum_{i<j} x_i x_j \left(x_i^2 + x_j^2\right) \leq C \left(\sum_i x_i\right)^4$$

holds for any $x_1, x_2, \ldots, x_n \geq 0$. For this constant C, determine when equality is achieved.

Problem 1.1.10. (Kvant, 1990) Prove that for any positive a, b, c no bigger than 1 holds

$$\frac{a}{bc+1} + \frac{b}{ca+1} + \frac{c}{ab+1} \leq 2.$$

Problem 1.1.11. (Bulgarian MO, 1997) Prove that for any positive a, b, c with $abc = 1$ holds

$$\frac{1}{1+a+b} + \frac{1}{1+b+c} + \frac{1}{1+c+a} \leq \frac{1}{2+a} + \frac{1}{2+b} + \frac{1}{2+c}.$$

Problem 1.1.12. (Singapore Training, 2009) If $x_1, x_2, \ldots, x_n$ are positive real numbers and $n \geq 5$, show that

$$\frac{x_1^3}{x_1^3 + x_2 x_3 x_4} + \frac{x_2^3}{x_2^3 + x_3 x_4 x_5} + \cdots + \frac{x_n^3}{x_n^3 + x_1 x_2 x_3} \leq n - 1.$$

1.2 Polynomials

Arguably one of the most powerful tools at your disposal is the good knowledge of polynomials and their properties. Let us start with a formal definition.

Definition 1.2.1. A polynomial $P(x)$ of degree n is a function of x of the type

$$P(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_0,$$

where the numbers a_i are elements of a field, e.g. $\mathbb{R}, \mathbb{Q}, \mathbb{Z}$ or $\mathbb{Z}_p$ and $a_0 \neq 0$.

One can look at a polynomial from the point of a linear combination of the monoms $1, x, x^2, \ldots$ and ascribe some combinatorial meaning to $a_k x^k$. We can think of it as a product of linear terms the way we think of an integer and its divisor. Or we can thinkg of it as a real valued differentiable function. Plynomials help bring combinatorics, algebra, geometry, and number theory together and are an invaluable piece of our arsenal.

Proposition 1.2.1. (Fundamental theorem of algebra) Every polynomial with complex coefficients of degree n has n complex root (counting multiplicity.) In other words, any polynomial $p(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n$ equals

$$a_0(x - x_1)(x - x_2)\cdots(x - x_n)$$

for some complex (not necessarily distinct) $x_1, x_2, \ldots, x_n$.

Proposition 1.2.2. If a polynomial $p(x)$ of degree no more than n takes the same value c for $n+1$ different x_i, $i = 1, 2, \ldots, n+1$, then $p(x) \equiv c$.

Proposition 1.2.3. Partial differences reduce the degree of a polynomial by 1, that is, $d_1(x) = p(x) - p(x-1)$ is of degree $\deg(p) - 1$. In general, if we define recurrently $d_k(x) = d_{k-1}(x) - d_{k-1}(x-1)$, then

$$d_k(x) = p(x) - \binom{k}{1}p(x-1) + \binom{k}{2}p(x-2) - \cdots + (-1)^k p(x-k)$$

is of degree $\deg(p) - k$ and has the same senior coefficient as $p^{(k)}(x)$.

This result can be used to prove an interesting result about polynomials with real coefficients who take integer values for integer x.

Proposition 1.2.4. If $p(x)$ is a polynomial of degree n with real coefficients and if $p(k) \in \mathbb{Z}$ for all integer k, then $p(x)$ can be uniquely represented as

$$p(x) = a_0 + a_1\binom{x}{1} + a_2\binom{x}{2} + \cdots + a_n\binom{x}{n},$$

where $a_0, a_1, \ldots, a_n$ are *integers*.

Proposition 1.2.5. (Viet's formulas) For every polynomial $p(x) = x^n + a_1 x^{n-1} + \cdots + a_n$ with roots $x_1, x_2, \ldots, x_n$ (not necessarily distinct) hold the following identities

$$\begin{aligned}
a_1 &= -(x_1 + x_2 + \cdots + x_n)\\
a_2 &= x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n\\
\ldots &= \ldots\\
a_k &= (-1)^k \sum_{0<i_1<i_2<\cdots<i_k\le n} x_{i_1}x_{i_2}\cdots x_{i_k}\\
\ldots &= \ldots\\
a_n &= (-1)^n x_1x_2\cdots x_n
\end{aligned}$$

A very important observations is that there is a unique polynomial of degree not exceeding $n-1$ and passing through any given n points $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ for some $x_1 < x_2 < \dots < x_n$. The following clumsy formula, more important for its claim of existence rather than for the exat polynomial, shows how this polynomial can be found.

Proposition 1.2.6. (Newton's interpolation formula) For any given real $x_1 < x_2, \dots, x_n$ and real or complex $y_1, y_2, \dots, y_n$, the polynomial

$$p(x) = \sum_{i=1}^{n} \prod_{j \neq i} \frac{x - x_j}{y_i - y_j} y_i$$

satisfies $p(x_i) = y_i$ for $i = 1, 2, \dots, n$ and $\deg p(x) \leq n - 1$.

The fact that this polynomial is unique is a direct consequence of proposition 1.2.2.

1.2.1 Integer polynomials and reducibility

We might wonder if an integer polynomial can be decomposed into integer polynomials of lower degree. Such property is called *reducibility.* We can speak of reducibility not only over $\mathbb{Z}$, but also over any other field, e.g. $\mathbb{Z}_p$, where we simply consider the coefficients modulo p. Here is a more formal definition.

Definition 1.2.2. A polynomial with integer coefficients $p(x)$ of degree $n > 1$ is called *irreducible over* $\mathbb{Z}$ if there do not exist polynomials $q(x)$ and $r(x)$ such that $p(x) = q(x)r(x)$. Similarly, a polynomial is irreducible over $\mathbb{Z}_p$ when $p(x) \equiv q(x)r(x) \pmod{p}$ has no solution.

The following claim is the backbone of all proofs of irreducibility.

Proposition 1.2.7. If $p(x)$ and $q(x)$ are integer polynomials, $q(x)$ with senior coefficient 1, then there exist integer polynomials $a(x)$ and $b(x)$ such that $p(x) = a(x)q(x) + b(x)$ and $\deg(b) < \deg(q)$.

Proposition 1.2.8. (Gauss'es lemma)

- Irreducibility over $\mathbb{Z}_p$ implies irreducibility over $\mathbb{Z}$.
- Irreducibility over $\mathbb{Z}$ implies irreducibility over $\mathbb{Q}$.

A useful property when dealing with polynomials over $\mathbb{Z}_{\not\vDash}$ is the following:

Proposition 1.2.9. For any integer polynomial $p(x)$ holds $p^2(x) \equiv p(x^2) \pmod{2}$.

Proposition 1.2.10. (Eisenstein's criterion) If $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ is a polynomial with integer coefficients and if there exists a prime p such that

(i) p divides a_i for $i = 0, 1, \ldots, n-1$;

(ii) p does not divide a_n;

(iii) p^2 does not divide a_0,

then $f(x)$ is irreducible.

Notice that if $p(x)$ is reducible(irreducible), then so is $p(x+1)$, so this trick can sometimes be used in conjunction with Eisenstein's criterion to prove irreducibility.

Corollary 1.2.1. *The polynomial* $1 + x + x^2 + \cdots + x^{p-1}$ *is irreducible over* $\mathbb{Z}$ *for any prime* p.

As a side note, we can talk about irreducibility not only over fields, but over any set, as demonstraded in problem 1.2.11.

A very powerful technique for proving irreducibility is analysis of the roots of a polynomial. For example, if 0 is not a root and only one of the complex roots has modulus bigger than or equal to 1, irreducibility follows from the fact that the rest of the complex roots cannot multiply up to an integer. See problem 1.2.1 for example.

1.2.2 Cyclotomic polynomials

The polynomial $x^n - 1$ can be decomposed into irreducible polynomials similarly to the way n can be decomposed into primes. The "primes" in the world of polynomials are called cyclotomic polynomials.

Definition 1.2.3. For a given n, let $\omega = e^{\frac{2\pi i}{n}}$. Then the polynomial

$$\Phi_n(x) = \prod_{(k,n)=1} (x - \omega^k)$$

is called *n-th cyclotomic polynomial*.

Proposition 1.2.11. For every n, $\Phi_n(x)$ has integer coefficients, has degree $\phi(n)$, and is irreducible over $\mathbb{Z}$.

Proposition 1.2.12. For every positive integer n holds

$$x^n - 1 = \prod_{d|n} \Phi_d(x).$$

Proposition 1.2.13. (Dirichlet's theorem) For every integer a there exist infinitely many n such that $an + 1$ is prime.

1.2.3 Chebyshev's Polynomials

Definition 1.2.4. A polynomial $T_n(x)$ that satisfies $T_n(\cos(\alpha)) = \cos(n\alpha)$ for all α is called *Chebyshev's Polynomial of order* n.

Proposition 1.2.14. Chebyshev's polynomials can be defined recurrently as follows:

- $T_0(x) = 1$ and $T_1(x) = x$;
- $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ for all $n \geq 1$.

Proposition 1.2.15. $T_n(x) \in [-1, 1]$ for all $x \in [-1, 1]$

The polynomials $T_2(x) = 2x^2 - 1$ and $T_3(x) = 4x^3 - 3x$ in particular appear often, yet sometimes in a "hidden" form, in many problems.

1.2.4 Exercises

Problem 1.2.1. (Singapore Training, 2009) Prove that the polynomial

$$p(x) = x^n + 5x^{n-1} + 3$$

is irreducible for all $n > 1$.

Problem 1.2.2. (Leningrad MO, 1991) A sequence $a_0, a_1, a_2, \ldots, a_{n-1}$ is called p-balanced if all sums $a_k + a_{k+p} + a_{k+2p} + a_{k+3p} + \cdots$ are equal for $k = 0, 1, \ldots, p-1$. Prove that if a sequence with 50 terms is p-blanaced for $p = 3, 5, 7, 11, 13, 17$, then all its terms are equal to 0.

Proposition 1.2.16. (Russian MO, 1992) Given are $2n$ different numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$. The entries of a given $n \times n$ table are filled according to the rule: put $a_i + b_j$ in the intersection of the ith row and the jth column. For each column, compute the product of the numbers in it. Suppose that those products are the same. Prove that the products of the numbers in each row are also the same.

Problem 1.2.3. (USAMO) Points $A_1, A_2, \ldots, A_n$ are picked on a circle with radius 1 so that for any point M on the circle holds

$$|MA_1| \cdot |MA_2| \cdots |MA_n| \leq 2.$$

Prove that $A_1, A_2, \ldots, A_n$ are vertices of a regular n-gon.

Problem 1.2.4. (Romanian MO, 2001) Find all pairs (m, n) of positive integers, such that m divides $a^n - 1$ for $a = 1, 2, \ldots, n$

Problem 1.2.5. (Polish MO, 1995) Let $p \geq 3$ be a given prime. Define the sequence $a_1, a_2, \ldots$ by $a_n = n$ for $n = 0, 1, \ldots, p-1$, and $a_n = a_{n-1} + a_{n-p}$ for $n > p$. Determine the remainder of a_{p^3} modulo p.

Problem 1.2.6. (IMO SL) Let $a_0, a_1, \dots, a_{n-1}$ be real numbers that sum up to 0. Prove that if the cyclic sum

$$\sum_C \frac{1}{a_i(a_i + a_{i+1}) \dots (a_i + a_{i+1} + \dots + a_{i+n-2})}$$

is well defined, then it is equal to 0.

Problem 1.2.7. (IMO SL, 1996) Let $P(x)$ be the real polynomial function, $P(x) = ax^3 + bx^2 + cx + d$. Prove that if $|P(x)| \leq 1$ for all $x \in [-1, 1]$ then

$$|a| + |b| + |c| + |d| \leq 7.$$

Problem 1.2.8. (IMO, 1988) Show that the solution set of the inequality

$$\sum_{k=1}^{70} \frac{k}{x-k} \geq \frac{5}{4}$$

is a union of disjoint intervals, the sum of whose lengths is 1988.

Problem 1.2.9. (IMO SL, 1996) Let n be an even positive integer. Prove that there exists a positive integer k such that

$$k = f(x) \cdot (x+1)^n + g(x) \cdot (x^n + 1)$$

for some polynomials $f(x), g(x)$ having integer coefficients. If k_0 denotes the least such k, determine k_0 as a function of n, i.e. show that $k_0 = 2^q$ where q is the odd integer determined by $n = q \cdot 2^r, r \in \mathbb{N}$.

Problem 1.2.10. (IMO, 2002) Find all pairs of positive integers $m, n \geq 3$ for which there exist infinitely many positive integers a such that

$$\frac{a^m + a - 1}{a^n + a^2 - 1}$$

is itself an integer.

Problem 1.2.11. Prove the polynomial $1 + x + x^2 + \dots + x^{p-1}$ is irreducible over $\mathbb{R}_0^+$ for any prime p, i.e. that it cannot be represented as the product of two non-constant polynomials with non-negative real coefficients.

1.3 Sequences

When dealing with sequences, it is sometimes necessary to change the form. Suppose $a_1, a_2, \dots$ is some sequence defined in some way. It helps to consider the following sequences

- $\{s_n\}, n \geq 1$ where $s_n = a_1 + a_2 + \dots + a_n$ are the partial sums (i.e. integrate the sequence)

- $\{d_n\}, n \geq 1$ where $d_1 = a_1$ and $d_n = a_n - a_{n-1}$ for $n > 1$ (i.e. differentiate the sequence)
- Put $b_n = \dfrac{1}{a_n}$ or some other transform that allows for an easier description of b_n.
- Define the sequence backward, that is, define a_0, a_{-1}, a_{-2} in a way that preserves the properties or the recurrent relationship.

The following is a very useful identity between partial sums and differences

Proposition 1.3.1. (Abel)

$$\begin{aligned} a_1b_1 + a_2b_2 + \cdots + a_nb_n =& (b_1 - b_2)a_1 + \\ & (b_2 - b_3)(a_1 + a_2) + \cdots + \\ & (b_{n-1} - b_n)(a_1 + a_2 + \cdots + a_{n-1}) + \\ & b_n(a_1 + a_2 + \cdots + a_n) \end{aligned}$$

1.3.1 Recurrent equations

Suppose that a sequence $a_1, a_2, a_3, \ldots$ is defined in the following way: the first few terms are given, and after that every term is a function of the previous ones. We shall say that the sequence is defined recurrently and shall call this function *recurrent equation*. For example, the sequence $\{a_n\}$ defined by $a_1 = 1$ and $a_n = a_{n-1} * n$ for $n \geq 2$. This is a simple example, since this sequence has a closed form, namely, $a_n = n!$. In general, such a form might not be available, yet there is still hope to prove some propertes of this sequence.

Sometimes, in order to prove a claim A (especially by induction,) we want to prove a stronger claim B. The same advice goes for iequalities, but there it is obvious anyway. The reason I mention it here is that a stronger claim yields more power in an inductive argument. For example, proving $a_n > \sqrt{n+1}$ might not be easy using induction but proving the stronger $a_n > 1 + \sqrt{n}$ might be doable.

1.3.2 Fibonacci

Consider the class of sequences where each term is a linear combination of a fixed number of preceding sequences, that is

$$a_n = k_1a_{n-1} + k_2a_{n-2} + \cdots + k_la_{n-l}$$

for all $n > l$ and where the first l terms determine the rest of the sequence. There are a number of interesting properties that such sequences posses, but since Fibonacci is the most famous example of such sequence, we can concentrate on it.

Definition 1.3.1. The sequence $F_1, F_2, \ldots$ defined as

- $F_1 = F_2 = 1$;
- $F_n = F_{n-1} + F_{n-2}$

is called Fibonacci's sequence.

Proposition 1.3.2. For every n,

$$F_n = \alpha a^n + \beta b^n,$$

where $(\alpha, \beta) = (\frac{\sqrt{5}}{5}, -\frac{\sqrt{5}}{5})$ and a and b are the roots of the polynomial $x^2 = x+1$.

Proposition 1.3.3. For every $n > 2$,

$$F_{n-1}^2 - F_n F_{n-2} = (-1)^n.$$

Proposition 1.3.4. For any prime p there exists n such that p divides F_n.

Proposition 1.3.5. For every n,

$$\frac{F_n}{F_{n+1}} = \underbrace{\frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\cdots + 1}}}}}_{n \text{ fractions}}.$$

Proposition 1.3.6. (Bulgarian MO, 1997) The number of non-empty subsets of $S_n = \{1, 2, \ldots, n\}$ which do not contain two consecutive numbers is $F_{n+2} - 1$.

Problem 1.3.1. (IMO SL, 1992) Let $\{x\}$ denote the fractional part of x. Pick any x_1 in $[0, 1)$ and define the sequence $x_1, x_2, \ldots$ by $x_{n+1} = 0$ if $x_n = 0$ and $x_{n+1} = \left\{ \frac{1}{x_n} \right\}$ otherwise. Prove that

$$x_1 + x_2 + \ldots + x_n < \frac{F_1}{F_2} + \frac{F_2}{F_3} + \ldots + \frac{F_n}{F_{n+1}},$$

where F_n is the n-th term of the Fibonacci sequence.

1.3.3 Binary representation

The following sequence $a_i, i \geq 0$, is defined recurrently as follows:

1. start with $a_0 = 0$;
2. on each turn, invert the terms (0 to 1 and 1 to 0) that have already been written and append it to the right.

The sequence starts with $0, 1|1, 0|1, 0, 0, 1|1, 0, 0, 1, 0, 1, 1, 0|1 \ldots$. What makes it special is not only its interesting definition, but also the following property.

Proposition 1.3.7. The number n has an odd number of ones in its binary representation iff a_n equals 1.

You shall meet this sequence unexpectedly in various situations.

1.3.4 The symbol {x}

Suppose that a is an *irrational* number and consider the sequence

$$a_n = \{na\}, \qquad n = 0, 1, 2, \ldots,$$

where $\{x\} = x - \lfloor x \rfloor$ denotes the *fractional part* of x. This sequence only takes values between 0 and 1 and the values form an arithmetic series modulo 1, that is, if we "solder" the ends of the interval $[0, 1]$ to form a circle. So we can think of the values of this series as a sequence of points $A_0, A_1, A_2, \ldots$ on a circle with circumference 1, defined as $\widehat{A_{i+1}A_i} = \widehat{A_iA_{i-1}}, (A_{i+1} \neq A_{i-1})$ for $i = 1, 2, 3, \ldots$.

Notice that these points never repeat, otherwise $\{m_1 a\} = \{m_2 a\}$ for some $m_1 \neq m_2$ would imply that $m_1 a$ and $m_2 a$ differ by an integer and hence that a is rational.

Proposition 1.3.8. For every $0 \leq x < y \leq 1$ there exist infinitely many n such that $x < \{na\} < y$.

Proposition 1.3.9. For every $\epsilon > 0$ and real $x_1, x_2, \ldots, x_k$, there exist infinitely many n such that for every integer $i \in [1, k]$ holds either $\{nx_i\} < \epsilon$ or $\{nx_i\} > 1 - \epsilon$.

1.3.5 Best approximation to 1

Define the sequence $a_n, n \geq 1$ as

(i) $a_1 = 2$;

(ii) $a_n = a_{n-1}^2 - a_{n-1} + 1$ for $n > 1$;

This sequence has a number of interesting properties.

Proposition 1.3.10. For every $n > 1$ holds $a_n = a_{n-1}a_{n-2} \cdots a_1 + 1$;

Proposition 1.3.11. For every n holds

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} + \frac{1}{a_{n+1} - 1} = 1.$$

Proposition 1.3.12. If $b_1, b_2, \ldots, b_n$ are positive integers such that

$$\sum_{i=1}^{n} \frac{1}{b_i} < 1,$$

then

$$\sum_{i=1}^{n} \frac{1}{b_i} \leq \sum_{i=1}^{n} \frac{1}{a_i}$$

with equality holding iff $b_1, b_2, \ldots, b_n$ are some permutation of $a_1, a_2, \ldots, a_n$. To prove, assume the contrary and use Abel's transform.

1.3.6 Catalan

A series $C_0, C_1, \ldots$ defined as

(i) $C_0 = 1$;

(ii) $C_n = C_0C_{n-1} + C_1C_{n-2} + \cdots + C_{n-1}C_0$ for all $n \geq 1$.

is called series of Catalan and can be explicitly written as

$$C_n = \binom{2n}{n} - \binom{2n}{n-1} = \frac{\binom{2n}{n}}{n+1}$$

Catalan's series, $(1, 1, 2, 5, 11, \ldots)$ appears wherever the recurrent equation above appears, which is the natural result of induction in many cases. Consider for example the following propositions.

Proposition 1.3.13. Show that the number of possible triangulations of a convex n-gon is equal to C_{n-2} (i.e. the $n-2$-nd term of Catalan's series.)

Proposition 1.3.14. Consider all pathes $(x_0, y_0), (x_1, y_1), \ldots (x_{2n}, y_{2n})$ in a coordinate system starting at $(0, 0)$ and ending at $(2n, 0)$, such that $x_{i+1} = x_i+1$ and $y_{i+1} = y_i \pm 1$ for $i = 0, 1, \ldots, 2n-1$. Prove that the number of those pathes that *do not* cross the x-axis (i.e. the ones where $y_i \geq 0$ for all i) is equal to C_n.

There are multiple ways to prove the formula above, the most popular being generating functions (see the relevant section in the text) and reflection across the x-axis in the reformulation of Catalan's series in proposition (1.3.14).

1.3.7 Exercises

Problem 1.3.2. (Bulgarian MO, 1995) Let x and y be two different real numbers. Prove that if $\dfrac{x^n - y^n}{x - y}$ is integer for four consecutive integer then it is integer for all positive integer n.

Problem 1.3.3. (Russian Olympiad, 1978 and IMO, 1991) Construct a bounded infinite sequence $x_0, x_1, x_2, \ldots$ such that

$$|x_i - x_j||i - j| \geq 1$$

for every pair of distinct nonnegative integers i, j.

Problem 1.3.4. (IMO, 1980) Define the numbers $a_0, a_1, \ldots$ in the following way:

$$a_0 = \frac{1}{2}, \quad a_{k+1} = a_k + \frac{a_k^2}{n} \quad (n > 1, k = 0, 1, \ldots, n-1).$$

Prove that for all n holds

$$1 - \frac{1}{n} < a_n < 1.$$

Problem 1.3.5. For a given real a, prove that there exist infinitely many integers x and $y > 0$ such that

$$\left| a - \frac{x}{y} \right| < \frac{1}{y^2}.$$

Problem 1.3.6. (Kvant 1978) The sequence $c_1, c_2, \ldots$ is defined as $c_1 = 2$ and $c_{n+1} = \left\lfloor \frac{3c_n}{2} \right\rfloor$ for all $n \geq 1$. Prove that

(i) There are infinitely many odd and infinitely mane even numbers in this sequence;

(ii) The sequence $a_n = (-1)^{c_n}$ is not periodic;

(iii) There exists a number γ such that $c_n = \left\lfloor \left(\frac{3}{2}\right)^n \gamma \right\rfloor + 1$ for every n.

Proposition 1.3.15. Given two postivie numbers a and b, the sets A and B are defined as $A = \{\lfloor na \rfloor : n \in \mathbb{N}\}$ and $B = \{\lfloor nb \rfloor : n \in \mathbb{N}\}$. We say that A and B *partition* $\mathbb{N}$ if $A \cup B = \mathbb{N}$ and $A \cap B = \emptyset$. Prove that A and B partition $\mathbb{N}$ iff a and b are irrational numbers satisfying $a + b = ab$.

Problem 1.3.7. For a given n, find the number of sequences of *non-negative integers* $a_0, a_1, a_1, \ldots, a_{2n}$ which satisfy:

(i) $a_0 = a_{2n} = 0$;

(ii) $a_i - a_{i-1} = \pm 1$ for $i = 1, 2, \ldots, 2n$.

Problem 1.3.8. (IMO 1982)

(i) Given is an non-increasing sequence $1 = x_0, x_1, x_2, \ldots$. Prove that there exists n such that $\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \frac{x_2^2}{x_3} + \cdots + \frac{x_{n-1}^2}{x_n} \geq 3.9999$;

(ii) Prove that there exists a non-increasing sequence $1 = x_0, x_1, x_2, \ldots$ such that, for any n, $\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \frac{x_2^2}{x_3} + \cdots + \frac{x_{n-1}^2}{x_n} < 4$.

1.4 Analysis

It might sound ridiculous to those who went to college that we put section Analysis under Algebra, but since none of you, readers, went to college and your brains are still intact, we shall merily assume that analysis is part of algebra.

In one word, real analysis deals with analyzing real valued functions. Consider such a function and let us inquire about its properties:

- How does it behave on a given interval? Does it decrease, stay constant, jump up and down, spike, or is it not entirely defined?

- How does it behave when x goes to infinity? Does it go to infinity itself, and if yes, how fast?
- How steep is it at a given point?

Answering such questions is precisely the subject of Analysis.

1.4.1 Derivatives

My assumption is that you are already familiar with derivatives, but since this is an important concept, I will run the risk of boring you by going quickly through the basics.

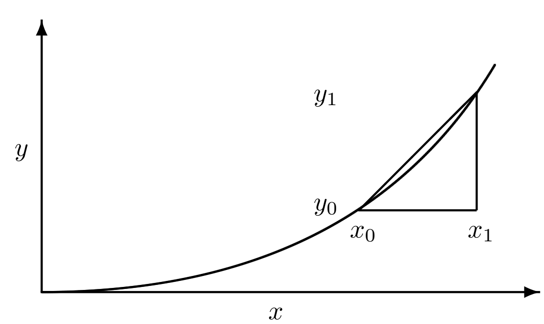
Consider the function $f(x) = x^2$ drawn below. It increases "faster and faster" the larger x becomes. The way to quantify the speed of increasing is to consider the slope at a given point (x_0, y_0). We might take $x_1 \approx x_0$ and $y_1 = f(x_1)$ and consider the ratio

$$\frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y}{\Delta x},$$

but this will depend on the proximity of x_1 to x_0 and on whether x_1 is higher or lower than x_0 since $f(x) = x^2$ has curvature. The right way to compute the instantaneous slope of f is to consider the limit

$$\lim_{x_1 \to x_0} \frac{y_1 - y_0}{x_1 - x_0},$$

and if this limit happens to exist, no matter how x_1 approaches x_0, then this limit will be the slope of f at x_0 and we shall denote it by $f'(x_0)$. If we can compute it for all x_0 in some interval, we can construct the function $f'(x)$ which tells us how $f(x)$ behaves – whether it increases ($f' > 0$,) decreases ($f' < 0$,) stays constant ($f' = 0$,) or maybe increases at a constant rate ($f' = c$ for some constant c.)



1.4.2 Convergence

Consider a series $a_1, a_2, \ldots$ and the question: "does the a_i *converge* to any number"? The question is natural, but before we can answer it for any sequence, we need a stricter definition of convergence.

Definition 1.4.1. We shall say that $a_1, a_2, \ldots$ converges to c if for any $\epsilon > 0$ there is a number n_0 such that $|a_n - c| < \epsilon$ for all $n > n_0$ and shall denote convergence in the following way:

$$\lim_{n\to\infty} a_n = c.$$

This definition might sound confusing, so here is what it *really* means: c is such a number that no matter how small a circle you draw around it, after some point, all a_i move inside it and never escape from there. In the above definition, a_i might be real as well as complex, and the definition could be naturally be expanded to allow $c = \pm\infty$.

Here are some criteria for convergence.

Proposition 1.4.1. If $a_1, a_2, \ldots$ is a *bounded monotonic sequence* (either non-decreasing or non-increasing and bounded between two constants,) then a_i converges.

We might wonder about convergence of a series, that is, an infinite sum.

Definition 1.4.2. We shall say that the *series* $a_1 + a_2 + \cdots$ converges if the *partial sums* $s_n = a_1 + a_2 + \cdots + a_n$ converge.

Proposition 1.4.2. If $\lim_{n\to\infty} a_n = 0$, then the series $a_1 - a_2 + a_3 - a_4 + \cdots$ converges.

Proposition 1.4.3. The following series converge

- $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots;$
- $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} \cdots;$
- $1 + 2a + 3a^2 + 4a^3 + \cdots$ for all $|a| < 1$;
- $1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ for all x;
- $\sum_{n\geq 0} \frac{\binom{2n}{n}}{(n+1)4^n};$

The followong series diverge (i.e. do not converge)

- $1 + \frac{1}{2} + \frac{1}{3} + \cdots;$

Problem 1.4.1. (IMO, 1985) For every real number x_1, construct the sequence $x_1, x_2, \ldots$ by setting:

$$x_{n+1} = x_n \left(x_n + \frac{1}{n} \right).$$

Prove that there exists exactly one value of x_1 which gives $0 < x_n < x_{n+1} < 1$ for all n.

1.5 Functional equations

Often times, a function is given and something is asked about its properties. *Conversely*, functional equations are problems where some properties of an unknown in question function are listed. Problems of this kind generally sound like: *Find all functions $f : A \to B$ that satisfy some conditions.*

1.5.1 Basic techniques

There are a number of techniques that can set the ground for a solution.

The first thing to do is to identify some general properties of $f : A \to B$.

injective Different points in A map to diferent values in B, i.e. $f(x) = f(y)$ implies $x = y$.

surjective(onto) Every value in B is the image of some point in A, i.e. for any $y \in B$ there exists $x \in A$ such that $f(x) = y$.

bijective(one-to-one) A function that is *both* injective and surjective.

Next, try to play with the properties by putting certain values for x or y, setting $x =$, swapping x and

- If possible, try putting $x = 0$ or some other value, or try $x = x_0$ s.t. $f(x_0) = 0$ when f is surjective. The purpose is to clear as many unwanted terms as possible;
- Try replacing substituting $f(x)$ or a similar expression for x, especially when $f(f(x))$ is known to be "nice".
- Try swapping x and y when one of the sides of the equation is symmetric.
- Try fixing x or y and varying the other variable. Once you discover some nice property, substitute back in the previous equation (before having fixed x or y).
- If $f_0(x)$ is an obvious solution, consider $g(x) = f(x) - f_0(x)$ or $g(x) = \dfrac{f(x)}{f_0(x)}$ and prove that $g(x) \equiv 0$ or $g(x) \equiv 1$ respectively.
- Try to find a fixed point, that is, x_0 such that $f(x_0) = x_0$.

Definition 1.5.1 (Cauchy's functions)**.** Functions $f : \mathbb{Q} \to \mathbb{Q}$ satisfying the following equation are called Cauchy's functions:

$$f(x+y) = f(x) + f(y).$$

Apparently, all functions of the type $f(x) = ax$ for a given integer constant a are Cauchy's functions.

1.5.2 Exercises

Problem 1.5.1. (Winter Competition, Bulgaria, 1996) Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$3f(x) - 2f(f(x)) = x$$

for all integer x.

Problem 1.5.2. (Tournament of Towns, 1997) Prove that there does not exist a function $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(x)) = x^2 - 1997$ for every x.

Problem 1.5.3. (IMO SL, 2000) Find all pairs of functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $f(x + g(y)) = xf(y) - yf(x) + g(x)$ for all real x, y.

Problem 1.5.4. (Vietnam MO, 1999) Let $S = \{0, 1, 2, \ldots, 1999\}$ and $T = \{0, 1, 2, \ldots\}$. Find all functions $f : T \to S$ such that

(i) $f(s) = s$ for all $s \in S$;

(ii) $f(m+n) = f(f(m) + f(n))$ for all $m, n \in T$.

Problem 1.5.5. (Balkan MO, 2000) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(xf(x) + f(y)) = f^2(x) + y$$

for all $x, y \in \mathbb{R}$.

Problem 1.5.6. (IMO SL, 1996) Let f be a function from the set of real numbers $\mathbb{R}$ into itself such for all real x we have $|f(x)| \leq 1$ and

$$f\left(x + \frac{13}{42}\right) + f(x) = f\left(x + \frac{1}{6}\right) + f\left(x + \frac{1}{7}\right).$$

Prove that f is a periodic function (that is, there exists a non-zero real number c such that $f(x+c) = f(x)$ for all real x.)

Problem 1.5.7. (IMO, 1977)(*) Let $f : \mathbb{N} \to \mathbb{N}$ be a function satisfying the inequality $f(n+1) > f(f(n))$ for all $n \in \mathbb{N}$. Prove that $f(n) = n$ for all n.

Problem 1.5.8. (IMO, 1982) The function $f(n)$ is defined on the positive integers and takes non-negative integer values. Also, $f(2) = 0, f(3) > 0, f(9999) = 3333$ and for all positive integers m, n holds

$$f(m+n) - f(m) - f(n) = 0 \text{ or } 1.$$

Determine $f(1982)$.

Problem 1.5.9. (IMO, 1983) Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$f(xf(y)) = yf(x)$$

for all $x, y \in \mathbb{R}^+$ and $f(x) \to 0$ as $x \to \infty$.

Problem 1.5.10. (IMO, 1986) Find all functions f defined on the non-negative reals and taking non-negative real values such that: $f(2) = 0$, $f(x) \neq 0$ for $0 \leq x < 2$, and for all $x, y \geq 0$

$$f(xf(y))f(y) = f(x + y).$$

Problem 1.5.11. (IMO, 1987) Prove that there is no function f from the set of non-negative integers into itself such that $f(f(n)) = n + 1987$ for all n.

Problem 1.5.12. (IMO, 1990) Let $\mathbb{Q}^+$ be the set of positive rational numbers. Construct a function $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that

$$f(x(f(y)) = \frac{f(x)}{y}$$

for all x, y in $\mathbb{Q}^+$.

Problem 1.5.13. (IMO, 1992) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\left(x^2 + f(y)\right) = y + (f(x))^2 \qquad \text{for all } x, y \in \mathbb{R}.$$

Problem 1.5.14. (IMO, 1993) Determine if there exists a strictly increasing function $f : \mathbb{N} \to \mathbb{N}$ with the following properties:

(i) $f(1) = 2$;

(ii) $f(f(n)) = f(n) + n$ for all $n \in \mathbb{N}$.

Problem 1.5.15. (IMO, 1994) Let S be the set of all real numbers strictly greater than -1. Find all functions $f : S \to S$ satisfying:

(i) $f(x + f(y) + xf(y)) = y + f(x) + yf(x)$ for all x, y in S;

(ii) $\dfrac{f(x)}{x}$ is strictly increasing on the intervals $-1 < x < 0$ and $0 < x$.

Problem 1.5.16. (IMO, 1996) Let $\mathbb{N}_0$ denote the set of nonnegative integers. Find all functions $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that

$$f(m + f(n)) = f(f(m)) + f(n) \qquad \text{for all } m, n \in \mathbb{N}_0.$$

Problem 1.5.17. (IMO, 1998) Determine the least possible value of $f(1998)$ where f is a function from the set $\mathbb{N}$ of positive integers into itself such that for all $m, n \in \mathbb{N}$

$$f\left(n^2 f(m)\right) = m\,(f(n))^2.$$

Problem 1.5.18. (IMO, 1999) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$$

for all $x, y \in \mathbb{R}$.

Problem 1.5.19. (IMO, 2002) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$(f(x) + f(z))\,(f(y) + f(t)) = f(xy - zt) + f(xt + yz)$$

for all real x, y, z, t.

1.5.3 Euler's formula

Euler's formula and the hyperbolic functions have little direct applications in IMO problems, but a competitor must know about them. It is not impossible for one to come up with an original solution using the concept of hyperbolic functions. Here we present the basics in brief.

Consider the expansion

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots .$$

The series on the right converges not only for all real z but also for any complex number too. This makes it quite convenient to define e^{ix}, and from there the extention of the definition to a^z – raizing a positive number to a complex degree – becomes quite natural.

Definition 1.5.2. For every $x \in \mathbb{R}$, define e^{ix} as follows:

$$e^{ix} = 1 + \frac{ix}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \cdots .$$

Proposition 1.5.1. (Euler's formula) For every $x \in \mathbb{R}$ holds

$$e^{ix} = \cos(x) + i\sin(x).$$

Definition 1.5.3. *Hyperbolic* cosine and sine are defined as

$$\cosh(x) := \frac{e^x + e^{-x}}{2} = \cos(ix)$$
$$\sinh(x) := \frac{e^x - e^{-x}}{2} = -i\sin(ix)$$

Proposition 1.5.2. (Pythagorean Theorem)

$$\cosh^2(x) - \sinh^2(x) = \cos^2(x) + \sin^2(x) = 1.$$

1.6 Complex numbers

Complex numbers are defined as a linear combination of 1 and i in real numbers, that is $x + yi$, and can thus be looked at as 2-dimensional vectors. Complex numbers *addition* is defined as vector (or a point in the xy-plane) and *multiplication* is defined by stipulating $i^2 = -1$.

Since a complex number z is a point in the xy-plane, it can be described through its real and imaginary parts (x and yi respectively), or by its distance from 0 and direction with respect to the real line:

$$z = x + yi = r(\cos(\theta) + i\sin(\theta)),$$

where $r = \sqrt{x^2 + y^2}$ and $\tan(\theta) = \dfrac{y}{x}$ when $x \neq 0$. The number r is referred to as *modulus* and θ is called *argument*. The common notation is $r = |z|$ and $\theta = \arg(z)$.

Proposition 1.6.1. When complex numbers are multiplied, their arguments "add" and their moduli "multiply", that is:

- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$;
- $|z_1 z_2| = |z_1||z_2|$.

Definition 1.6.1. The *conjugate* of a complex number $z = a + bi$ is defined as the number

$$\overline{z} = a - bi.$$

Proposition 1.6.2. For any complex z_1 and z_2, the following identities hold

- $|\bar{z_1}| = |z_1|$;
- $\bar{z_1}\bar{z_2} = \overline{z_1 z_2}$;
- $\bar{z_1} + \bar{z_2} = \overline{z_1 + z_2}$.

In competitions, we use complex numbers either in geometry (usually as a last resort) or in combinatorics (usually the roots of 1). They have their nice applications in certain number theory problems as well, but no such problems seem to appear on IMO.

Chapter 2

Geometry

2.1 Basics

2.1.1 Triangle and quadrilateral

Proposition 2.1.1. For a given triangle $\triangle ABC$, put $a = BC, b = AC$, and $\gamma = \angle ACB$. Then

$$S_{\triangle ABC} = \frac{ab\sin(\gamma)}{2}.$$

Proposition 2.1.2. For a given convex quadrilateral $ABCD$, let the diagonals have lengths a and b and let the angle between them be γ. Then

$$S_{ABCD} = \frac{ab\sin(\gamma)}{2}.$$

Theorem 2.1.1. *(Pascal's Theorem) Let $A_1, A_2, \dots, A_6$ lie on a circle*[1]*. Prove that the following points(, intersections of the segments or their continuations,) are collinear:*

$$P = A_1A_3 \cap A_2A_6$$
$$Q = A_4A_6 \cap A_3A_5$$
$$R = A_1A_4 \cap A_2A_5$$

Theorem 2.1.2. *(Menelaus' theorem) Given are a triangle $\triangle ABC$ and three points A_1, B_1, C_1 lying on the lines BC, AC and AB respectively. The three points A_1, B_1, C_1 are collinear iff*

$$\frac{BA_1}{A_1C} \cdot \frac{CB_1}{B_1A} \cdot \frac{AC_1}{C_1B} = -1,$$

where the ratios $\dfrac{XV}{VY}$ are to be taken with sign $+$ or $-$ depending on whether V lies inside or outside the segment XY.

[1]The theorem is valid for an ellipse as well

Theorem 2.1.3. *(Ceva's Theorem) Given are a triangle $\triangle ABC$ and three points A_1, B_1, C_1 lying on the lines BC, AC and AB respectively. The lines AA_1, BB_1 and CC_1 intersect at a point iff*

$$\frac{BA_1}{A_1C} \cdot \frac{CB_1}{B_1A} \cdot \frac{AC_1}{C_1B} = 1,$$

where the ratios $\dfrac{XV}{VY}$ are to be taken with sign $+$ or $-$ depending on whether V lies inside or outside the segment XY.

Proposition 2.1.3. (Ceva's Theorem – sine form) Given are a triangle $\triangle ABC$ and three points A_1, B_1, C_1 lying on the lines BC, AC and AB respectively. The lines AA_1, BB_1 and CC_1 intersect at a point iff

$$\frac{\sin(\angle BAA_1)}{\sin(\angle A_1AC)} \cdot \frac{\sin(\angle CBB_1)}{\sin(\angle B_1BA)} \cdot \frac{\sin(\angle ACC_1)}{\sin(\angle C_1CB)} = 1,$$

where the angles are *directed*.

Proposition 2.1.4. (Carnot's theorem) Let A_1, B_1, C_1 lie on the sides (or their continuations) BC, AC, AB of a triangle $\triangle ABC$. Prove that the perpendiculars to BC, AC, AB through A_1, B_1, C_1 respectively meet at a point iff

$$AB_1^2 - B_1C^2 + BC_1^2 - C_1A^2 + CB_1^2 - B_1A^2 = 0.$$

Proposition 2.1.5. If AB and CD intersect at a point M (either external or internal to AB and CD at the same time,) then $MA \cdot MB = MC \cdot MD$ if and only if A, B, C, D are concyclic.

Proposition 2.1.6. If the altitude from C meets the circumcircle of $\triangle ABC$ secondarily at C_1, then C_1 is symmetric to H with respect to AB; as a consequence, the circumcircles of $\triangle ABH$ and $\triangle ABC$ have the same radius.

Proposition 2.1.7. If the incircle of $\triangle ABC$ touches AB at point T, then $AT = \dfrac{AB + AC - BC}{2}$.

2.1.2 Vectors and barycentric coordinates

For our purposes, we shall think of vectors as n-tuples of real numbers (i.e. elements of $\mathbb{R}^n$) and shall define several operations in the vector space. If $\vec{a} = (a_1, a_2, \ldots, a_n), a_i \in \mathbb{R}$ and $\vec{b} = (b_1, b_2, \ldots, b_n), b_i \in \mathbb{R}$ are any two vectors in $\mathbb{R}^n$ and λ is a real number, we shall define

Vector addition $\vec{a} + \vec{b} = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$;

Multiplication with a number $\lambda\vec{a} = (\lambda a_1, \lambda a_2, \ldots, \lambda a_n)$;

Dot product $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + \cdots + a_nb_n$.

Notice that the dot product is a number, not a vector. Geometrically, we can think of the vector $\vec{a}$ as the *set of all directed segments* $\overrightarrow{XY}$ in $\mathbb{R}^n$, where the coordinates $x_1, x_2, \ldots, x_n$ of the endpoint X and $y_1, y_2, \ldots, y_n$ of the point Y satisfy $y_i - x_i = a_i$ for $i = 1, 2, \ldots, n$.
The *lenght* of a vector is defined as $|\vec{a}| = \sqrt{\vec{a} \cdot \vec{a}}$, which is a positive number for any vector except the *zero vector*.

Proposition 2.1.8. For any non-zero vectors $\vec{a}$ and $\vec{b}$ holds

$$-1 \leq \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} \leq 1$$

with equality on the right(left) holding when there exists a positive(negative) number λ such that $\vec{a} = \lambda\vec{b}$.

Definition 2.1.1. The *angle* between $\vec{a}$ and $\vec{b}$ is the (only) number $\phi \in [0, \pi]$ such that $\cos(\phi) = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$.

Proposition 2.1.9. In $\mathbb{R}^2$, the angle between two vectors (defined above) is the same as the geometrical angle between their directed segments. In other words, if α denotes the angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$, then

$$\overrightarrow{AB} \cdot \overrightarrow{CD} = AB \cdot CD \cos(\alpha).$$

Proposition 2.1.10. (Cosine theorem) For any triangle $\triangle ABC$,

$$AC^2 + AB^2 - 2AC \cdot AB \cos(\angle CAB) = BC^2.$$

Proposition 2.1.11. Given n points $M_1, M_2, \ldots, M_n$ in space, the point that minimizes the sum $MM_1^2 + MM_2^2 + \cdots + MM_n^2$ is the center of gravity of $M_1, M_2, \ldots, M_n$. In particular, the point M in the plane that minimizes $MA^2 + MB^2 + MC^2$ for a given triangle $\triangle ABC$ is its centroid.

When we use vectors, we often want to express the condition that a point P lies on a line AB. In vector notation, that is equivalent to finding a real λ such that $\vec{OP} = \lambda\vec{OA} + (1 - \lambda)\vec{OB}$. The following proposition helps us classify the location of P with respect to the points A and B.

Proposition 2.1.12. If for some $\lambda \in \mathbb{R}$ and points P, A, B holds $\vec{OP} = \lambda\vec{OA} + (1 - \lambda)\vec{OB}$, then P lies on the line AB and

- P is between A and B iff $\lambda \in (0, 1)$;
- P is outside AB and closer to A if $\lambda > 1$;
- P coincides with A(or B) when $\lambda = 1$ (or $\lambda = 0$.)

The notion of *barycentric coordinates* extends this idea. For each point P in the plane of $\triangle ABC$, consider linear combination $x\overrightarrow{PA}+y\overrightarrow{PB}+z\overrightarrow{PC}$. There is at least one triplet that makes this combination equal to $\vec{0}$, ant that is the trivial $x=y=z=0$. Since this is of little information, we consider triplets where at least one of x, y, and z is not 0. If (x, y, z) is such, then aparently $(3x, 3y, 3z)$ and $(-x, -y, -z)$ also works, so we shall deem them the same. We call (x, y, z) *homogeneous* coordinates, where by homogeneos we mean that coordinates that are proportional to each other are virtually the same.

Definition 2.1.2. The homogeneous coordinates (x, y, z) for which

$$x\overrightarrow{PA}+y\overrightarrow{PB}+z\overrightarrow{PC}=\vec{0}$$

are called *barycentric coordinates* of P with respect to $\triangle ABC$.

Proposition 2.1.13. For each point P and triangle $\triangle ABC$, there are unique barycentric coordinates of P with respect to $\triangle ABC$, which *do not* sum up to zero. The barrycentric coordinates of

- the vertices A, B, and C are $(1,0,0),(0,1,0)$, and $(0,0,1)$ respectively;
- all points on the side AB are $(x, 1-x, 0)$ for some $x \in [0,1]$.
- all points inside the triangle $\triangle ABC$ are positive;
- the centroid are $(1,1,1)$;
- the orthocenter are $(\tan(\alpha), \tan(\beta), \tan(\gamma))$;
- the circumcenter are $(\sin(2\alpha), \sin(2\beta), \sin(2\gamma))$;
- the incircle are $(\sin(\alpha), \sin(\beta), \sin(\gamma))$.

The trick to determining barycentric coordinates is based on the following observation.

Proposition 2.1.14. If P has barycentric coordinates (x, y, z) with respect to $\triangle ABC$, then the following ratio holds for the *signed* areas below

$$S_{\triangle PAB} : S_{\triangle PBC} : S_{\triangle PCA} = x : y : z,$$

where the sign of $S_{\triangle XYZ}$ is defined as $+$ if X, Y, Z are in the same cyclical direction as A, B, C (clockwise or counterclockwise) and sign $-$ otherwise. In particular, for points P inside $\triangle ABC$, all areas above are taken with sign $+$.

You are advised to run through the statements in proposition 2.1.13 and see why they hold using the area principle. The reason we might want to use barycentric coordinates in practice is that we can use them as a criteria for whether three points are collinear.

Proposition 2.1.15. Given a triangle $\triangle ABC$ and points X, Y, Z with barycentric coordinates (x_1, x_2, x_3),(y_1, y_2, y_3), and (z_1, z_2, z_3) with respect to $\triangle ABC$ respecively, thre three points are collinear iff

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = 0.$$

Since the barycentric coordinates never sum up to zero, we can always consider the triple which sums up to 1 (since barycentric coordinates are homogeneous.) In that case, the above determinant equals the ratio of $S_{\triangle XYZ}$ and $S_{\triangle ABC}$, and its sign determines whether the orientation of $\triangle XYZ$ and $\triangle ABC$ is the same.

2.1.3 Pole, polar line, and radical axis

These are useful concepts with one or two useful properties each, which we list here.

Definition 2.1.3. Let k be a circle and M be a point external to k. If the tangets from P to k meet it at points M and N, then the line MN is called *polar line* of P, and the point P is called a *pole* of MN.

Theorem 2.1.4. *For any circle k and external points M and N, the polar line of M contains N iff the polar line of N contains M.*

Proposition 2.1.16. Given is a circle Γ and four points A, B, C, and D on it (in any order). If $P = AB \cap CD$ and $Q = AD \cap BC$, then P lies on the polar line of Q with respect to Γ and similarly Q lies on the polar line of P.

Definition 2.1.4. The *power* of a point M with respect to a circle k is the number $p = MA \cdot MB$, where $A, B \in k$ and $M \in AB$. From proposition 2.1.5 we know that it does not matter which exactly pair A, B we choose.

Definition 2.1.5. For any two non-concentric circles, the *radical axis* is defined as the locus of all points that have the same power with respect to each circle.

Theorem 2.1.5. *The radical axis of any two non-concentric circles is a line perpendicular to the line through their centers. If the two circles happen to intersect, then the radical axis passes through the intersection points.*

Proposition 2.1.17. For any three non-concentric circles k_1, k_2, and k_3, the three radical axes of k_1 and k_2, k_2 and k_3, and k_3 and k_1 intersect at a point.

2.1.4 Exercises

Problem 2.1.1. Let B_1 and C_1 be points on the sides AC and AB of $\triangle ABC$. The straight lines BB_1 and CC_1 intersect at point D. Prove that the quadrilateral AB_1DC_1 is circumscribed (in which a circle can be inscribed) iff the incircles of $\triangle ABD$ and $\triangle ACD$ are tangent.

Problem 2.1.2. (Kvant, 1979) There are five points on a circle and for every pair of points A and B among them a line perpendicular to AB and passing through the centroid of the other three points is built. Prove that all 10 lines pass through the same point.

Problem 2.1.3. (Kvant, 1999) A point inside an equilateral triangle is connected with the vertices of the triangle and with the feet of the perpendiculars from it to the sides. Thus, the triangle is split into 6 rectangular triangles, which are painted in red and blue alternatingly. Prove that the sum of the inradii of the blue triangles equals that of the red ones.

Problem 2.1.4. The diagonals AC and BD of a quadrilateral $ABCD$ are perpendicular if and only if $AB^2 + CD^2 = AD^2 + BC^2$.

Problem 2.1.5. Four squares with centers O_1, O_2, O_3 and O_4 are built ixternally on the sides of a convex quadrilateral $ABCD$. Prove that $O_1O_3 \perp O_2O_4$.

Problem 2.1.6. Given is a convex quadrilateral $ABCD$ and points P, Q, R, and S on AB, BC, CD, and DA respectively such that

$$\frac{AP}{PB} = \frac{DR}{RC} = \lambda \text{ and } \frac{AS}{SD} = \frac{BQ}{QC} = \mu.$$

If U is the intersection of PR and QS, prove that

$$\frac{SU}{UQ} = \lambda \text{ and } \frac{PU}{UR} = \mu.$$

Problem 2.1.7. Given is a quadrilateral $A_0B_0C_0D_0$ and points $A_1, \ldots, A_{n-1}$ on A_0B_0 such that $A_0A_1 = A_1A_2 = \cdots = A_{n-1}B_0$ for some *even* n. The points B_i, C_i, and D_i are defined analogously for $i = 1, 2, \ldots, n$. The lines A_iC_{n-i} and B_iD_{n-i} divide $A_0B_0C_0D_0$ into n^2 small quadrilaterals wchich are painted alternatingly in black and white like the squares of a chessboard. Prove that the total area of the white squares equals that of the black squares.

Problem 2.1.8. (Winter Competition, Bulgaria, 2001) Points A_1, B_1, and C_1 are chosen on the sides of BC, CA, and AB of a triangle $\triangle ABC$. Let G be the centroid of $\triangle ABC$, and G_a, G_b, and G_c be the centroids of $\triangle AB_1C_1$, $\triangle BA_1C_1$, and $\triangle CA_1B_1$. The centroids of $\triangle A_1B_1C_1$ and $\triangle G_aG_bG_c$ are denoted by G_1 and G_2 respectively. Prove that:

(i) G, G_1, and G_2 are collinear;

(ii) The lines AG_a, BG_b, and CG_c intersect at a point iff AA_1,BB_1, and CC_1 intersect at a point.

Problem 2.1.9. (Bulgarian MO, 1995) Points A_1, B_1, C_1 are chosen on the sides BC, AC, AB of $\triangle ABC$ respectively so that AA_1, BB_1, and CC_1 meet at a point M. Prove that if M is the centroid of $\triangle A_1B_1C_1$ then M is also the centroid of $\triangle ABC$.

Problem 2.1.10. Let P be a point inside the acute-angled $\triangle ABC$. Let X, Y, Z be external to $\triangle ABC$ such that $PBXC, PCYA, PAZB$ are concyclic quadrilaterals with perpendicular diagonals. If $PB \perp XZ$ and $PA \perp YZ$, prove that P is the circumcenter of $\triangle ABC$.

Problem 2.1.11. (IMO, 2008) Let H be the orthocenter of an acute-angled triangle $\triangle ABC$. The circle Γ_A centered at the midpoint of BC and passing through H intersects the sideline BC at points A_1 and A_2. Similarly, define the points B_1, B_2, C_1, and C_2. Prove that the six points A_1, A_2, B_1, B_2, C_1, and C_2 are concyclic.

Problem 2.1.12. Let the incircle Γ of a triangle $\triangle ABC$ with $\angle A = 60°$ meet AB, BC, and CA at C_1, A_1, and B_1 respectively. Define $P = \Gamma \cap AA_1, P \neq A_1$ and $T \in B_1C_1$ such that $APTI$ is concyclic. Find $\angle TA_1C_1$.

Problem 2.1.13. (Winter Competition, Bulgaria, 1996) The incenter of a quadrilateral $ABCD$ is denoted as O. The lines $l_A \perp OA, l_B \perp OB, l_C \perp OC$, and $l_D \perp OD$ are drawn through the points A, B, C, and D respectively. Let $K = l_A \cap l_B, L = l_B \cap l_C, M = l_C \cap l_D, N = l_D \cap l_A$.

(i) Prove that KM and LN meet at O.

(ii) If $OK = p, OL = q, OM = r$, find the length of ON.

Problem 2.1.14. (Winter Competition, Bulgaria, 1997) Let H be the orthocenter of an acute triangle $\triangle ABC$. Prove that the midpoints of AB and CH and the point of intersection of the internalbisectors of $\angle CAH$ and $\angle CBH$ lie on a line.

Problem 2.1.15. (Spring Competition, Bulgaria, 2001) Given is $\triangle ABC$ and points M, N, P interior to the angles $\angle BAC, \angle ABC$, and $\angle ACB$ respectively such that

$$\angle MAB = \angle MCA \text{ and } \angle MAC = \angle MBA$$
$$\angle NBA = \angle NCB \text{ and } \angle NBC = \angle NAB$$
$$\angle PCA = \angle PBC \text{ and } \angle PCB = \angle PAC$$

Prove that the lines AM, BN, and CP meet at a point on the circumcircle of $\triangle MNP$.

Problem 2.1.16. (IMO SL, 1998) Let ABC be a triangle, H its orthocenter, O its circumcenter, and R its circumradius. Let D be the reflection of the point A across the line BC, let E be the reflection of the point B across the line AC, and let F be the reflection of the point C across the line AB. Prove that the points D, E, and F are collinear if and only if $OH = 2R$.

Problem 2.1.17. The diagonals of a conves quadrilateral $ABCD$ divide it into four triangles with incenters I_1, I_2, I_3, and I_4. Prove that I_1, I_2, I_3, I_4 are concyclic *iff* $ABCD$ can be *outscribed* around a circle.

2.2 Trigonometry

For those of you who (just like me) are devoid of any geometrical creativity, the only way to master geometry and do well on competitions is to perfect trigonometry. In doing so, almost every geometrical problem reduces to a 3-page calculation and it is solely up to your speed to finish the problem within the given time.

2.2.1 Formulas

Here is a list of trigonometric formulas that hold for all real α and β. These are the bread and butter of the non-creative geometrists and must be learned by heart. Note that they easily derive from one another using the simple $\sin(\alpha) = \sin(\pi - \alpha) = \cos(\frac{\pi}{2} - \alpha)$; still, you must have each one of them ready to fire.

$$\begin{aligned}
\sin(\alpha \pm \beta) &= \sin(\alpha)\cos(\beta) \pm \sin(\beta)\cos(\alpha) \\
\cos(\alpha \pm \beta) &= \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta) \\
\sin^2(\alpha) &= \frac{1 - \cos(2\alpha)}{2} \\
\cos^2(\alpha) &= \frac{1 + \cos(2\alpha)}{2} \\
\sin(\alpha) \pm \sin(\beta) &= 2\sin\left(\frac{\alpha \pm \beta}{2}\right)\cos\left(\frac{\alpha \mp \beta}{2}\right) \\
\cos(\alpha) + \cos(\beta) &= 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right) \\
\cos(\alpha) - \cos(\beta) &= 2\sin\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\beta - \alpha}{2}\right) \\
\tan(\alpha + \beta) &= \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)} \\
\cot(\alpha + \beta) &= \frac{\cot(\alpha)\cot(\beta) - 1}{\cot(\alpha) + \cot(\beta)}
\end{aligned}$$

If r denotes the radius of the incircle, R of the circumcircle, S is the area, p is the semi-perimeter, and $2\alpha, 2\beta, 2\gamma$ are the angles of $\triangle ABC$ with sides a, b, c, then

$$\begin{aligned}
r &= 4R\sin(\alpha)\sin(\beta)\sin(\gamma) \\
S &= \sqrt{p(p-a)(p-b)(p-c)} \\
2R = \frac{a}{\sin(\alpha)} &= \frac{b}{\sin(\beta)} = \frac{c}{\sin(\gamma)} \\
\tan(2\alpha) + \tan(2\beta) + \tan(2\gamma) &= \tan(2\alpha)\tan(2\beta)\tan(2\gamma) \\
\sin(2\alpha) + \sin(2\beta) + \sin(2\gamma) &= 4\cos(\alpha)\cos(\beta)\cos(\gamma)
\end{aligned}$$

In the process of wild computations, make sure that you veryfy your calculations on *every line* by setting $\alpha = \beta = \gamma = 30°$ if the case permits. This way, the left and the right side become very easy to compute and that can in turn help you avoid a potential error in your calculation. The importance of this point cannod be stressed enough. The same advice goes for calculations with complex numbers.

2.2.2 Exercises

Problem 2.2.1. (Winter Competition, Bulgaria, 1997) Given is a triangle $\triangle ABC$ such that $\angle ABC \geq 60°$ and $\angle BAC \geq 60°$. Let $BL, (L \in AC)$ be the internal bisector of $\angle ABC$ and $AH, (H \in BC)$ be the altitude from A. Find $\angle AHL$ if $\angle BLC = 3\angle AHL$.

Problem 2.2.2. The externally inscribed circles of $\triangle ABC$ to the sides AB, BC, and CA meet CA, CB, AB, AC, BC, and BA at C_1, C_2, A_1, A_2, B_1, and B_2 respectively. Let $P_C = A_1A_2 \cap B_1B_2$, $P_A = B_1B_2 \cap C_1C_2$, and $P_B = A_1A_2 \cap C_1C_2$. Prove that AP_A, BP_B, and CP_C meet at the circumcircle of $\triangle P_AP_BP_C$.

Problem 2.2.3. (Bulgarian MO, 1996) Given is a convex quadrilateral $ABCD$, for which $\angle ABC + \angle BCD < 180°$. The lines AB and CD meet at point E. Prove that $\angle ABC = \angle ADC$ iff

$$AC^2 = CD \cdot CE - AB \cdot AE.$$

Problem 2.2.4. (IMO SL, 1998) Let $\triangle ABC$ be such that $\angle ACB = 2\angle ABC$. Let D be the point on the side BC such that $CD = 2BD$. The segment AD is extended to E so that $AD = DE$. Prove that

$$\angle ECB + 180° = 2\angle EBC.$$

2.3 Complex numbers

Given a coordinate system, every point in the plane corresponds to a complex number called *affix* of such point. Since every geometrical object can thus be represented in terms of complex numbers, we can prove geometrical identities analytically. On top of that, orientation is not a problem for complex numbers the way it is for trigonometry, since numeric identities don't really care about orientation (e.g. on which side of the quadrilateral the opposite sides meet.)

Let us see how the basic geometrical properties, collinearity, and perpendicularity are expressed in terms of complex numbers.

Proposition 2.3.1. The points A, B, C lie on a line iff their affixes a, b, c satisfy

$$\frac{a-c}{b-c} = \frac{\bar{a}-\bar{c}}{\bar{b}-\bar{c}}.$$

Proposition 2.3.2. The line AB is perpendicular to CD iff the affixes a, b, c, d satisfy

$$\frac{a-b}{c-d} = -\frac{\bar{a}-\bar{b}}{\bar{c}-\bar{d}}.$$

Since conjugation makes things messy, it really helps to have as many points as possible on the unit circle, where $\bar{x} = \frac{1}{x}$ holds. In that sense, the first thing to do when setting up a problem to solve in complex numbers is to position our triangle $\triangle ABC$ in the unit circle. It might help to rotate in such a way that $a = 1$, which might further simplify things, but the downside is that we lose the symmetry among a, b, and c.

Determining the intersection of AB and CD means sloving a system of two equations, namely $Z \in AB$ and $Z \in CD$ with two unknowns, namely z and $\bar{z}$. Again, things look much nicer when A, B, C, D lie on the unit circle than when they are arbitrary points.

Proposition 2.3.3. If a, b, c, d are the affixes of A, B, C, D on the unit circle, then the affix of $Z = AB \cap CD$ satisfies

$$z = \frac{(a+b)cd - (c+d)ab}{cd - ab}.$$

It is also helpful to know the intersection of two lines tangent to the unit circle at points A and B in case the unit circle is the incircle of a given triangle (as opposed to the more common case when we inscribe our triangle inside the unit circle.)

Proposition 2.3.4. If the tangents from an external point Z to the unit circle meet it at A and B, then

$$z = \frac{2ab}{a+b}.$$

The last construction that occurs often enough to warrant its formula listed here is the foot of a perpendicular dropped from a point to a chord in the unit circle.

Proposition 2.3.5. If A is a random point and BC is a chord in the unit circle, then the affix of the foot of the altitude from A onto BC is

$$z = \frac{a+b+c-bc\bar{a}}{2}.$$

Proving collinearity with complex numbers is messy in general and relies on the first two propositions stated earlier, yet concurency (three lines at a point) is a bit neater. If we want to prove that three symmetrically defined lines meet at a point, all we have to do is cross two of the lines and show that the expression for the point of intersection is symmetric with respect to, say, a, b, c. Then automatically every two of the three lines would meet at the same point. It might come handy to use the symmetric polynomials when the computations become unwieldy:

$$\begin{aligned} p &= a + b + c \\ q &= ab + bc + ca \\ r &= abc \end{aligned}$$

A good example where that would come to work the proof of Pascal's Theorem (see 2.1.1).

Proposition 2.3.6. The following points have the following affixes

$$\begin{aligned} \text{circumcenter } O &: o = 0 \\ \text{centroid } G &: g = \frac{a+b+c}{3} \\ \text{orthocenter } H &: h = a + b + c \\ \text{center of Euler's circle } E &: e = \frac{a+b+c}{2} \\ \text{incircle } I &: \textit{messy} \end{aligned}$$

One hint is to never set the circumcircle as the unit circle when the incircle is present; instead, one should set the incircle as the unit circle and consider the triangle *outscribed* around it.

2.3.1 Exercises

Problem 2.3.1. (Singapore Training, 2009) Let $ABCD$ be a cyclic quadrilateral and let the points K, L, M, N be the midpoints of the sides AB, BC, CD, and DA respectively. Show that the orthocenters of the four triangles $\triangle AKN$, $\triangle BKL$, $\triangle CLM$, and $\triangle DMN$ are the vertices of a parallelogram.

Problem 2.3.2. (Butterfly theorem) Let M be the midpoint of a chord AB in a circle. Let the chords CD and EF meet at M and let FC and ED meet AB at P and Q respectively. Prove that $PM = QM$.

Problem 2.3.3. Given is a regular n-gon $A_0A_1A_2 \cdots A_{n-1}$ inscribed in a circle of radius 1. Prove that $A_0A_1 \cdot A_0A_2 \cdots A_0A_{n-1} = n$.

Problem 2.3.4. Let $ABCD$ be inscribed in a circle with center O and let $P = AB \cap CD$, $Q = AD \cap BC$, and $H = AC \cap BD$. Prove that $OH \perp PQ$.

Problem 2.3.5. (IMO SL, 1998) Let $ABCDEF$ be a convex hexagon such that $\angle B + \angle D + \angle F = 360°$ and

$$\frac{AB}{BC} \cdot \frac{CD}{DE} \cdot \frac{EF}{FA} = 1.$$

Prove that

$$\frac{BC}{CA} \cdot \frac{AE}{EF} \cdot \frac{FD}{DB} = 1.$$

Problem 2.3.6. (Bulgarian MO, 1996) The quadrilateral $ABCD$ is inscribed in a circle. The lines AB and CD meet at point E, and the diagonals AC and BD meet at F. The circumcircles of $\triangle AFD$ and $\triangle BFC$ meet again at point $H \neq F$. Prove that $\angle EHF = 90°$.

Problem 2.3.7. (IMO SL, 1998) Let ABC be a triangle with $\angle A = 90°$ and $\angle B < \angle C$. The tangent at A to its circumcircle ω meets the line BC at D. Let E be the reflection of A across BC, X the foot of the perpendicular from A to BE, and Y the midpoint of AX. Let the line BY meet ω again at Z. Prove that the line BD is tangent to the circumcircle of $\triangle ADZ$.

Problem 2.3.8. Let $\triangle ABC$ be a non-equilateral triangle with circumcenter O, center of gravity G, and orthocenter H, and let F be symmetric to O with respect to the midpoint of GH. Consider three points A_1, B_1, C_1 on the sides BC, CA, and AB respectively such that $\triangle A_1B_1C_1 \sim \triangle ABC$. The lines through A, B, C that are parallel to B_1C_1, C_1A_1, and A_1B_1 respectively intersect at C_2, A_2, B_2 respectively. Prove that A_1A_2, B_1B_2, and C_1C_2 intersect at a point S such that $SF - SO = GH$.

2.4 Various transforms

When a geometric problem looks too complicated, it might help to change the sketch by moving each point to a new place according to some rule in the hope that the new configuration will be easier to tackle.

Such a rule is called a *transform*, which is just a function defined on the plane. It simply takes a point as an argument and moves it to another point.

2.4.1 Linear transforms

A transformin $\mathbb{R}^n$, that is, a transform defined as $f(x) = Ax + b$ for all vecotrs (points) $x \in \mathbb{R}^n$ for some fixed matrix A and vector b is called *linear(affine)*. Not all linear transforms are of interest to us, but the following ones in $\mathbb{R}^2$ prove very helpful on occasion.

Definition 2.4.1. A transform $t(\vec{a})$ which sends point A to a new point A' such that $AA' = \vec{a}$ for some fixed $\vec{a}$ is called *translation*. In the complex plane, translation is given by a function $f(x) = x + a$.

Definition 2.4.2. A transform $h(O, \lambda)$ which sends any point A to a new point A' on the ray $OA^{\rightarrow}$ such that $OA' = \lambda OA$ is called *homothety* with center O and coefficient λ. In the complex plane, homothety is given by a function $f(x) = \lambda(x - c) + c$, where c is the affix of the point O and λ is a real number.

Definition 2.4.3. A transform $r(O, \alpha)$ which sends any point A to a new point A' such that $OA' = OA$ and the *directed angle* $\angle A'OA = \alpha$ is called *rotation* with center O and angle. Any rotation preserves lengths, angles, and incidence. In the complex plane, rotation is given by a function $f(x) = \omega(x - c) + c$ where c is the affix of the point O and $\omega = e^{i\alpha}$.

Problem 2.4.1. Angles and incidence (e.g. whether two lines cross at a point or not) are preserved under translation, homothety, and rotation. In addition, lengths are preserved under translation and rotation.

Problem 2.4.2. If $h_1(O_1, \lambda_1)$ and $h_2(O_2, \lambda_2)$ are homothetys, then $h = h_2 \circ h_1$ is also homothety with center $O \in O_1O_2$ and coefficient $\lambda = \lambda_1 \lambda_2$.

Problem 2.4.3. If $r_1(O_1, \alpha_1)$ and $r_2(O_2, \alpha_2)$ are rotations, then $r = r_2 \circ r_1$ is also a rotation with angle $\alpha = \alpha_1 + \alpha_2$.

2.4.2 Orthogonal projection

Given a configuration in the plance, we can project orthogonally the whole plane onto another plane. There are a number of nice propoerties of orthogonal projection: Notice that

- Incidence is preserved, in other words, objects that intersect also intersect in the projection;
- Ratios on a line are preserved, that is, if A, B, C are different and collinear, then their images A_1, B_1, C_1 are also different collinear and $AB : BC = A_1B_1 : B_1C_1$.

While orthogonal projection might be the most natural, we are not bound to projecting orthogonally – we can do the opposite and project onto a slanted plane. Why would we want to do that? For example, we might project a given triangle into an equilateral one, so any geometrical statement about areas or ratios on lines in a triangle need only be proven for equilateral triangles for example. Or we might project into a rectangular triangle.

For example, in order to prove Menelaus' theorem, we might project the given triangle into a different plane into a equilateral triangle preserving the ratios in which the points A_1, B_1, C_1 split the sides of the triangle. In an equilateral triangle, trigonometry becomes a song. Really powerful tool, right? Unfortunately, it all falls apart as soon as a circle is introduced, since circles transform into ugly elipses. Orthogonal projection is best used when there are only *lines*, *ratios*, and *areas* involved in a problem.

The following interesting fact is also useful for proving two distances are the same.

Proposition 2.4.1. Let l, m be two non-parallel lines, and let AB, CD be two arbitrary segments. If the orthogonal projections of AB and CD onto l are equal, and if the orthogonal projections of AB and CD onto m are equal, then $AB = CD$ and $AB \parallel CD$.

2.4.3 Inversion

Definition 2.4.4. A transform $i(O, r^2)$ in $\mathbb{R}^n$ which sends any point $A \neq O$ into a point A' on the ray $OA^{\rightarrow}$ such that $OA \cdot OA' = r^2$ is called *inversion with center O and radius r*.

Any inversion that sends A to A' sends A' back to A (hence the name *inversion*,) and the fixed points of the inversion form a sphere of radius r around O.

Proposition 2.4.2. Under inversion $i(O, r)$, lines passing through O stay in place and lines not passing through O transform into circles though O.

Proposition 2.4.3. Angles between lines(planes) and circles(spheres) are preserved under inversion.

Proposition 2.4.4. If M' and N' are the images of M and N under inversion $i(O, r^2)$, then

$$M'N' = r^2 \frac{MN}{OM \cdot ON}.$$

2.4.4 Isogonal points

Definition 2.4.5. For a given triangle ABC, two points P and Q are said to be isogonal if the angular bisectors to $\angle A, \angle B$, and $\angle C$ also bisect the angles $\angle PAQ, \angle PBQ$, and $\angle PCQ$ respectively.

Sine Ceva Theorem guarantees that every point has a unique isogonal image (which might be the same point as in the case of the incenter and the outcenters).

2.4.5 Exercises

Problem 2.4.4. The perpendicular from the center of a circle k to an external line l crosses k at D. Let P and Q be arbitrary points on l and let $P' = PD \cap k$ and $Q' = QD \cap k$ be different from D. Prove that the circumcenter of $\triangle P'Q'P$ lies on k iff $PQ = P'Q'$.

Problem 2.4.5. Prove Ptolemy's inequality (Proposition 2.6.1).

Problem 2.4.6. Prove Pascal's theorem (Theorem 2.1.1). (Hint: consider isogonal points.)

Problem 2.4.7. (Tournament of Towns, Spring 2000) The chords AC and BD of a circle with center O intersect at the point K. The points M and N are the circumcenters of $\triangle AKB$ and $\triangle CKD$. Prove that $OM = KN$.

Problem 2.4.8. (Bulgarian MO, 1999) The vertices A, B, C of $\triangle ABC$ lie on the sides B_1C_1, A_1C_1, and A_1B_1 of $\triangle A_1B_1C_1$ and

$$\angle ABC = \angle A_1B_1C_1$$
$$\angle BCA = \angle B_1C_1A_1$$
$$\angle CAB = \angle C_1A_1B_1$$

Prove that the orthocenters of $\triangle ABC$ and $\triangle A_1B_1C_1$ are equally distanced from the circumcenter of $\triangle ABC$.

Problem 2.4.9. (Tournament of Towns, Fall 2004) The rays $OA^{\to}, OC^{\to}, OB^{\to}$, and $OD^{\to}$ are drawn in this order so that $\angle AOB = \angle COD$. Circles inscribed in $\angle AOB$ and $\angle COD$ meet at points E and F. Prove that $\angle AOE = \angle DOF$.

Problem 2.4.10. Given are circles k and k' inside of k. It is known that there exist circles $k_0, k_1, k_2, \ldots, k_{n-1}$ such that k_i is tangent to k, k', k_{i-1}, and k_{i+1} for $i = 0, 1, \ldots, n-1$ (the indices are taken modulo n.) Let c_0, $c_1, \ldots$ be a series of circles such that c_i touches k, k', and c_{i-1} for $i > 0$. Prove that $c_n \equiv c_0$.

Problem 2.4.11. (Russian Olympiad, 2001) Given are a tetrahedron $SABC$ and a sphere through A, B and C whose center lies in the plane ABC and which intersects the edges SA, SB, and SC a second time in the points A_1, B_1, and C_1 respectively. The tangent planes to this sphere at points A_1, B_1, and C_1 intersect at O. Prove that O is the circumcentre of the tetrahedron $SA_1B_1C_1D_1$.

Problem 2.4.12. Given is $\triangle ABC$ with $\angle B = 60°$. The incircle of the triangle meets AB at T and $C_1 \neq C$ is such that T is the midpoint of CC_1. If the perpendicular bisector of BC_1 meets the angular bisector of $\angle A$ at A_1, prove that $\triangle A_1BC_1$ is equilateral.

Problem 2.4.13. Let P and Q be internal to the acute-angled $\triangle ABC$ so that

$$\angle PCA = \angle PAB = \angle PBC \quad \text{and} \quad \angle QAC = \angle QBA = \angle QCB.$$

Prove that

(i) the feet of the perpendiculars from P and Q onto the sides of $\triangle ABC$ lie on a circle centered at the midpoint of PQ;

(ii) if O is the circumcenter of $\triangle ABC$, then $PO = QO$.

Problem 2.4.14. (IMO SL, 1999) Two circles Ω_1 and Ω_2 touch internally the circle Ω in M and N and the center of Ω_2 is onΩ_1. The common chord of the circles Ω_1 and Ω_2 intersects Ω in A and B. MA and MB intersect Ω_1 in C and D. Prove that CD is tangent to Ω_2.

2.5 Special points and objects

2.5.1 Euler's line and Euler's circle

Theorem 2.5.1. *Given is a triangle $\triangle ABC$ with orthocenter H and circumcenter O. Let H_a, H_b and H_c denote the foots of the altitudes from A, B, and C respectively and A_1, B_1, and C_1 be the midpoints of BC, AC, and AB respectively. Then $A_1, A_2, A_3, H_a, H_b, H_c$ and the midpoints of HH_a, HH_b, and HH_c lie on a circle with center the midpoint of OH. This is the* circle of nine points *or else* Euler's circle.

Theorem 2.5.2. *If O, G, H, and E denote the circumcenter, the center of gravity, the orthocenter, and the center of Euler's circle defined above. Then O, G, E, and H line on a line, called* Euler's line *and* $OG : GE : EH = 2 : 1 : 3$.

Proposition 2.5.1. If E is the center of Euler's circle for $\triangle ABC$, then its projections on the sides of the triangle are collinear iff E lies on the circumcirlce of $\triangle ABC$.

Proposition 2.5.2. For any triangle, Euler's circle is tangent to its incircle and its externally inscribed circles.

2.5.2 Toricelli

Given a triangle $\triangle ABC$, we might ask what is the point M in the plane that minimizes $MA+MB+MC$. We know that $MA^2+MB^2+MC^2$ is minimized for $M = G$ (see proposition 2.1.11), but without the squares it becomes a different question with no nice answer for more than 3 points.

Definition 2.5.1. For a given acute angled triangle $\triangle ABC$, there is a unique point M inside $\triangle ABC$ such that $\angle AMB = \angle BMC = \angle CMA = 120°$. It is called point of *Toricelli*.

Proposition 2.5.3. The point that minimizes $MA + MB + MC$ for a given triangle $\triangle ABC$ is

- The point of Toticelli for $\triangle ABC$ if it is acute-angled;
- The vertex with the obtuse angle in case of an obtuse-angled $\triangle ABC$.

2.6 Geometrical inequalities

Proposition 2.6.1. (Ptolemy's inequality) For a convex quadrilateral $ABCD$,

$$AB \cdot CD + AD \cdot BC \geq AC \cdot BD.$$

Proposition 2.6.2. If R and r are the circumradius and inradius respectively, and d is the distance between the circumcenter and the incenter, then

$$d^2 = R(R - 2r),$$

which also implies that $R \geq 2r$ for any triangle.

2.6.1 Exercises

Problem 2.6.1. Prove that for any convex polygon and a point M inside it, there is at least one side of the polygon such that M projects orthogonally inside it.

Proposition 2.6.3. (Singapore Training, 2009) Let $ABCDEF$ be a cyclic hexagon with $AB = BC$, $CD = DE$, and $EF = FA$. Prove that

$$\frac{AB}{BE} + \frac{CD}{DA} + \frac{EF}{FC} \geq \frac{3}{2}.$$

Problem 2.6.2. (Kvant, 1983) The points A_1, B_1, and C_1 are picked on the sides BC, CA, and AB of $\triangle ABC$ such that AA_1, BB_1, and CC_1 intersect at a point. Prove that $S_{\triangle A_1B_1C_1} \leq S_{\triangle ABC}$.

Problem 2.6.3. (Kvant, 1983) Prove that in every convex polygon M one can inscribe a rectangle with area at least $\frac{1}{4}$ of the area of M.

Problem 2.6.4. A figure A in the plane is called *p-figure* if there exists a coordinate system and real numbers $a, b, c (a \neq 0)$ such that A is the locus of all points (x, y) such that $y \geq ax^2 + bx + c$. Is it possible to cover the whole plane with a finite number of p-figures?

Problem 2.6.5. (IMO, 1996) Let $ABCDEF$ be a convex hexagon such that AB is parallel to DE, BC is parallel to EF, and CD is parallel to FA. Let R_A, R_C, R_E denote the circumradii of $\triangle FAB, \triangle BCD, \triangle DEF$ respectively, and let P denote the perimeter of the hexagon. Prove that

$$R_A + R_C + R_E \geq \frac{P}{2}.$$

Problem 2.6.6. (IMO SL, 1996) Let $\triangle ABC$ be an acute triangle with circumcenter O and circumradius R. The radius OA meets the circumcircle of $\triangle BOC$ at A', BO meets the circumcircle of $\triangle COA$ at B', and CO meets the circumcircle of $\triangle AOB$ at C'. Prove that $OA' \cdot OB' \cdot OC' \geq 8R^3$.

Problem 2.6.7. (Spring Competition, Bulgaria, 1995) Given is a convex polygon with $n \geq 4$ vertices, no 4 of which lie on the sme circle. Prove that there exists a circle through three consecutive vertices of the polygon which contains the remaining vertices in its interior.

Problem 2.6.8. Given is a point O and a set $\mathcal{A}$ of $n > 2$ points $A_1, A_2, \ldots, A_n$ which is symmetric with respect to O. If no three points of $\mathcal{A}$ are collinear and $OA_1 + OA_2 + \cdots + OA_n > 7$, prove that there exist two different points $P, Q \in \mathcal{A}$ such that the circumradius of $\triangle OPQ$ is bigger than 1.

Chapter 3

Combinatorics

3.1 Set theory

Let's start with some notations. If A is a finite set, we use $|A|$ or $\#A$ to denote the number its elements. When A has infinitely many elements, we cannot talk about number of elements per se, but the notion of size set still exists. We call it *cardinality* of A.

Comparing two sets means finding an injective function from one to the other. For example, it is easy to define an injective function from $A = \{1, 2, 5\}$ to $B = \{a, b, c, d\}$ but not from B to A, impying that $|B| > |A|$. Finding an injective function from $f : \mathbb{Z} \to \mathbb{Q}$ is quite natural, for example $f(x) = x$; however, it might come as a surprise that an injection from $g : \mathbb{Q} \to \mathbb{Z}$ exists as well, therefore $|\mathbb{Z}| = |\mathbb{Q}| = |\mathbb{Z}|^2$. Keep in mind that $|A|$ is not an integer number when A is infinite.

Here are two must-know propositions.

Proposition 3.1.1. If there exists injective functions $f : A \to B$ and $g : B \to A$, then there exists a bijection between A and B.

Proposition 3.1.2. No injective function exists from $\mathbb{R}$ to $\mathbb{Q}$. In other words, $|\mathbb{R}| > |\mathbb{Q}|$.

3.2 Binomial coefficients

Binomial coefficients are the building bricks of combinatorics. They have posses combinatorical as well as algebraic properties which allows for proving algebraic equalities using counting or proving combinatorial facts algebraically.

Definition 3.2.1. For every n, the coefficients of the expansion of $(1 + x)^n$ are called *binomial coefficients* of order n. For $k = 0, 1, \ldots, n$, the coefficient in front of x^k is denoted as $\dbinom{n}{k}$.

Proposition 3.2.1. For all integer n, k with $n \geq k \geq 1$ holds

$$\binom{n}{n-k} = \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Proposition 3.2.2. Each binomial coefficient could be expressed analytically in the form

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Proposition 3.2.3. The sum of all coefficients of order n is 2^n, and the sum with alternating signs is 0, that is

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

Proposition 3.2.4. There are exactly $\binom{n}{k}$ sequences of n ones are zeroes containing k ones and $n-k$ zeroes.

3.2.1 Identities

Problem 3.2.1. Prove that for every positive integer n

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

Problem 3.2.2. Prove that for all $n \geq 0$

$$\sum_{k \geq 0} \binom{n-k}{k} = F_{n+1},$$

where F_n is the n-th term of Fibonacci's sequence.

Problem 3.2.3. Prove that for all $n > 0$,

$$\sum_{k>0} k \binom{n}{k} = n2^{n-1}.$$

Problem 3.2.4. (IMO SL, 1991) Prove that

$$\sum_{0 \leq k < n} \frac{(-1)^k}{n-k} \binom{n-k}{k} = \frac{(-1)^{n-1}}{n}.$$

Problem 3.2.5. Prove that $\sum_{k=1}^{n}(-1)^{n-k}\binom{n}{k}\binom{kn-1}{n-1} = 1$ for every positive integer n.

Problem 3.2.6. (Kvant, 1973) For every positive integer n prove the equalities:

(i) $\binom{n}{0} - \frac{\binom{n-1}{1}}{4} + \frac{\binom{n-2}{2}}{4^2} - \cdots + \frac{(-1)^k\binom{n-k}{k}}{4^k} + \cdots = \frac{n+1}{2^n}$, where the sum is take onver all non-negative integers k not exceeding $\frac{n}{2}$.

(ii) $\binom{n}{0} - pq\binom{n-1}{1} + p^2q^2\binom{n-2}{2} - \cdots + p^kq^k(-1)^k\binom{n-k}{k} + \cdots = \frac{p^{k+1}-q^{k+1}}{p-q}$, where $p+q=1$ and $p \neq q$, and the sum is take onver all non-negative integers k not exceeding $\frac{n}{2}$.

Problem 3.2.7. (Kvant, 1980) For every positive integer n prove the equalities:

(i) $\frac{1}{\binom{n}{0}} - \frac{1}{\binom{n}{1}} + \frac{1}{\binom{n}{2}} - \cdots + \frac{(-1)^k}{\binom{n}{k}} + \cdots + \frac{(-1)^n}{\binom{n}{n}} = \frac{n+1}{n+2}(1+(-1)^n)$.

(ii) $\frac{1}{\binom{n}{0}} + \frac{1}{\binom{n}{1}} + \frac{1}{\binom{n}{2}} + \cdots + \frac{1}{\binom{n}{k}} + \cdots + \frac{1}{\binom{n}{n}} = \frac{n+1}{2^{n+1}}\left(\frac{2}{1} + \frac{2^2}{2} + \frac{2^3}{3} + \cdots + \frac{2^n}{n}\right)$.

3.2.2 Pascal's triangle

Pascal's triangle is defined as a series of rows, each starting and ending with one and the rest of the $n-1$ numbers in row n defined as the sum of the numbers from the $n-1$th row right above them.

Strictly speaking, Pascal's tringle's elements in the n-th row could be defined as a series $P_{k,n}$ where $P_{0,n} = P_{n,n} = 1$ and $P_{k,n} = P_{k-1,n-1} + P_{k,n-1}$. Notice that $P_{i,k}$ satisfies the same recurrent formula 3.2.1 and starting conditions as the binomial coefficients, so we can conclude that $P_{k,n} = \binom{n}{k}$.

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

The interesting properties of Pascal's triangle are revealed when we consider it modulo p for some prime p. We need some armoury under our belt.

Proposition 3.2.5. (Luka's theorem) Given is a prime p. Let n and k, $k \in [0,n]$, have the representations

$$n = \overline{a_l a_{l-1} \cdots a_0} \text{ and } k = \overline{b_l b_{l-1} \cdots b_0}$$

in p-digit system. Then

$$\binom{n}{k} \equiv \binom{a_0}{b_0}\binom{a_1}{b_1}\cdots\binom{a_l}{b_l} \pmod{p}.$$

Consider for example $p = 3$ and Pascal's triangle modulo p

```
            /1\
           /1 1\
          /1 2 1\
        /1\ 0 0 /1\
       /1 1\ 0 /1 1\
      /1 2 1\ /1 2 1\
    /1\ 0 0 /2\ 0 0 /1\
   /1 1\ 0 /2 2\ 0 /1 1\
  /1 2 1\ /2 1 2\ /1 2 1\
 /1\ 0 0 0 0 0 0 0 0 /1\
/1 1\ 0 0 0 0 0 0 0 /1 1\
```

3.2.3 Exercises

Problem 3.2.8. (IMO, 1993) Given are $n > 1$ lamps $L_0, L_1, \ldots, L_{n-1}$ in a circle, all of which are initially ON. On step $j = 0, 1, 2, \ldots$, lamp L_j is switched (from ON to OFF or vice versa) iff L_{j-1} is ON (the subscripts are taken mod n). Prove that

(i) Sooner or later after the first step, all lamps will be ON again;

(ii) If $n = 2^k$ for some integer k, then all the lamps will be ON after $n^2 - 1$ steps;

(iii) If $n = 2^k + 1$ for some integer k, then all lamps will be ON after $n^2 - n + 1$ steps.

Problem 3.2.9. For a prime $p > 3$, consider p integers arranged on a circle. On each turn, we add at the same time to each number its right neighbor and subtracted twice its left neighbor. Prove that after $p - 1$ turns all the numbers will be congruent modulo p

Proposition 3.2.6. (Kvant, 1970) Prove that for any integer $n \geq 2$, 2^{3n} divides

$$\binom{2^{n+1}}{2^n} - \binom{2^n}{2^{n-1}},$$

but 2^{3n+1} does not.

Problem 3.2.10. Define the sequence of polynomials $p_0(x), p_1(x), \ldots,$ as follows: $p_0(x) = x$ and $p_n(x) = 1 + p_0(x)p_1(x)\cdots p_{n-1}(x)$ for all $n > 0$. Prove that for $n > 0$, the number of the odd coefficients of $p_n(x)$ equals $2^k + 1$ for some integer k.

3.3 Counting

Half of combinatorics is counting objects that have some property. For example, the number of choosing k people among n or the number of permutations of order n etc.

Most countinig problems have four faces:

- combinatorial formulation, e.g. number of permutations of $1, 2, \ldots, n$;
- formula, e.g. $n!$;
- reccurent relationship, e.g. $a_n = na_{n-1}$;
- polynomial (generating function), e.g. $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$.

They usually ask you to find the link between those different faces. For example, one problem might ask you to fing the *formula* expressing the number of permutations of $1, 2, \ldots, n$. Sometimes, the link is not direct. You might be given a recurent formula and asked to find the closed form formula, which would entail designing a combinatorial statement or a polynomial in the process.

Some are quite fundamental and appear often in one form or another, so we list them as propositions. Try to describe each of the four faces for each proposition.

Proposition 3.3.1. The number of the non-decreasing sequences of k integers between 1 and n is $\binom{n+k-1}{k}$.

3.3.1 General stuff

Problem 3.3.1. (Euler) In how many ways can a postman deliver n letters to n different recipients so that not a single recipient receive his own letter?

Problem 3.3.2. Given is the sequence $a_1, a_2, \ldots, a_n$ of different real numbers for which is known that there does not exist a decreasing subsequence of length more than k. Show that the number of pairs $i < j$ for which $a_i > a_j$ does not exceed $\frac{k-1}{2k}n^2$.

Problem 3.3.3. (Bulgarian MO, 1995) Let $n > 1$ be an integer. Find the number of all permutations $(a_1, a_2, \ldots, a_n)$ of $1, 2, \ldots, n$ such that there exists a single index $i \in [1, n-1]$ such that $a_i > a_{i+1}$.

Problem 3.3.4. (IMO, 1989) Let n and k be positive integers and let S be a set of n points in the plane such that

(i) no three points of S are collinear; and

(ii) for every point P of S there are at least k points of S equidistant from P.

Prove that $k < \frac{1}{2} + \sqrt{2n}$.

Problem 3.3.5. In a tennis tournament of n players, each player i played against everyone else scoring a_i wins and b_i losses. Show that

$$a_1^2 + a_2^2 + \cdots + a_n^2 = b_1^2 + b_2^2 + \cdots + b_n^2.$$

Problem 3.3.6. We shall say that x *belongs to* $((a,b))$ if $x \neq b$ and $(x-a)(x-b)(a-b) \geq 0$. A partitioning of the set $\{1, 2, 3, \ldots, 2n\}$ into ordered couples $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$ is called *special* if the following conditions hold:

(i) There is an integer $p \in [1, 2n]$ which belongs to $((a_i, b_i))$ for all i;

(ii) There is an integer $q \in [1, 2n]$ which does not belong to $((a_i, b_i))$ for any i.

Prove that there are exactly $2n \cdot n!$ special partitionings.

Problem 3.3.7. A person took an n-dollar loan and has to repay it in multiple instalments (not necessarily the same amount every time). In how many ways can the loan be payd out if

(i) the number of instalments is k?

(ii) the number of instalments is not fixed?

Problem 3.3.8. (IMO SL, 1999) Let n be a positive integer. A *path* from $(0,0)$ to (n,n) in the xy-plane is a chain of consecutive moves either to the right (move denoted by E) or upwards (move denoted by N,) all the moves being made inside the halfplane $x \geq y$. A *step* in a path is the occurence of two consecutive moves of the form EN.
Show that the number of paths from $(0,0)$ to (n,n) that contains exactly s steps $(n \geq s \geq 1)$ is

$$\frac{1}{s}\binom{n-1}{s-1}\binom{n}{s-1}.$$

Problem 3.3.9. (Winter Competition, Bulgaria, 1997) A table with 100 rows and 1997 columns is filled with zeroes and ones in such a way that every column contains at least 75 ones. Prove that 95 rows can be removed so that at most one of the columns in the remaining table contains zeroes only.

Problem 3.3.10. (Winter Competition, Bulgaria, 2001) The plane is divided into unit squares by lines parallel to the coordinate axes of an orthogonal coordinate system. Find the number of paths of length n from the point $(0,0)$ to the point (a,b) moving along the sides of the unit squares.

Problem 3.3.11. (Tournament of Towns, Autumn 2000) The little squares of an m by n table are painted in two colors. It is known that if a rook is placed on any square, it will attack less squares of the same color (that of the square it is placed on) than squares of the other color. (A rook attacks all the squares in the row and in the column it is placed on, including the one it stands on.) Prove that in each row and in each column the number of squares of one color is the same as that of the other one.

Problem 3.3.12. (Tower of Hanoi) Given are n disks of different size stacked on a rod in order of size (the smallest being on top) and two other empty rods. On each move, one is permitted to move a disk from one rod to another unless placing a larger disk onto a smaller one. What is the minimal number of moves required to move all the disks from one rod to another?

Problem 3.3.13. (Tournament of Towns, Spring 2000) In a chess tournament, each participant plays against everyone else exactly once. A win is worth one point, a draw is worth half a point, a loss is zero points. Let us call a game an *upset* if the player who won has gained fewer[1] points in the total score than the player who lost.

(i) Prove that no matter what the results of the tournament, strictly less than $\frac{3}{4}$ of the games were upsets.

(ii) Prove that one cannot replace the number $\frac{3}{4}$ in (i) by a smaller number.

Problem 3.3.14. (IMO, 1997) For each positive integer n, let $f(n)$ denote the number of ways of representing n as a sum of powers of 2 with nonnegative integer exponents. Representations which differ only in the ordering of their summands are considered to be the same. For instance, $f(4) = 4$, because the number 4 can be represented in the following four ways: 4; 2+2; 2+1+1; 1+1+1+1.
Prove that, for any integer n we have $2^{\frac{n^2}{4}} < f(2n) < 2^{\frac{n^2}{2}}$.

Problem 3.3.15. (IMO, 1998) In a contest, there are m candidates and n judges, where $n \geq 3$ is an odd integer. Each candidate is evaluated by each judge as either pass or fail. Suppose that each pair of judges agrees on at most k candidates. Prove that

$$\frac{k}{m} \geq \frac{n-1}{2n}.$$

3.3.2 Vectors

We introduced vectors as orderd n-tuples of real numbers (which is a definition to cause a heart stroke to any mathematician but is good enough for us).They have very nice use in combnatorics. The first typical application has its foundation in the following fact:

Theorem 3.3.1. *For every $n+1$ vectors $a_1, a_2, \ldots, a_{n+1}$ in $\mathbb{R}^n$ there exist real numbers $\lambda_1, \lambda_2, \ldots, \lambda_{n+1}$,* not all zero*, such that*

$$\sum_{i=1}^{n+1} \lambda_i a_i = \vec{0}.$$

[1]Maybe they meant "no more than" instead of "fewer"

Here are a few examples using that “linear dependency“ theorem.

Problem 3.3.16. (Russian Olympiad, 2001) At a competition, n multi-choice problems were given. Each problem $i \in [1, n]$, brought in $p_i \in \mathbb{N}$ points for a correct answer an 0 for an incorrect one. After the contestants submitted their answer sheets, they were ranked according to their total score.
The jury noticed that any other ranking of contestants would have been possible by choosing some other set of numbers $p_1, p_2, \ldots, p_n$. What is the maximum possible number of contestants at this competition?

Problem 3.3.17. For some prime p and positive integer k, the surface of a cylinder with circumference p and k has been divided into $p \times k$ unit “squares”. A real number is written in each square so that the following is true: for each square, the sum of the numbers in the squares with which it shares a side is 0. If at least one of the pk numbers is different from zero, prove that $k \geq p - 1$.

Sometimes it just helps to translate a combinatorial problem into the language of linear algebra and, say, the field of remainders modulo 2.

Problem 3.3.18. (Russian Olympiad, 1972) At the end of a one-round hockey tournament, it turned out that for every group of teams one could find a team (possibly from the same group) which earned an *odd* number of points playing the teams of this group. Prove that the number of all teams participating in the tournament is *even.* (A loss brings 0 points, a draw earns one point to each team, and a victory brings 2 points).

Sometimes, the trick consists of counting *pairs* of objects as opposed to objects themselves. That brings us to the other use of vectors, namely when it comes to counting intersections of objects. If a set A of a subset of n elements is represented as a n-element vector of 1s and 0s (1 for every element of S that is in A and 0 for any element that is not in A), say vector a, then the number of elements in $A \cap B$ is exactly $a \cdot b$. [put some examples here]

3.3.3 Summation of sets

Problem 3.3.19. Prove that among any $2n-1$ integers one can choose n whose sum is divisible by n.

Problem 3.3.20. Let d be the least integer such that $n|2^d + 1$. If $n > 6d$, prove that there exist more than $6d$ integers with *different* remainders modulo n which have no more than 3 ones in their binary representation.

Problem 3.3.21. (Bulgarian Olympiad, 2004) Let p be a prime number and let $0 \leq a_1 < a_2 < \cdots < a_m < p$ and $0 \leq b_1 < b_2 < \cdots < b_n < p$ be arbitrary integers. Let k be the number of distinct residues modulo p that $a_i + b_j$ give when i runs from 1 to m, and j from 1 to n. Prove that

(i) if then $m + n > p$, then $k = p$;

(ii) if $m + n \leq p$, then $k \geq m + n - 1$.

Problem 3.3.22. (IMO SL, 2003)(*) Let p be a prime number and let A be a set of positive integers that satisfies the following conditions

(i) the set of prime divisors of the elements in A consists of $p-1$ elements;

(ii) for any nonempty subset of A, the product of its elements is not a perfect p-th power.

What is the largest possible number of elements in A?

3.3.4 Inclusion/exclusion principle

Given are n objects and k properties. Every object might posses some, all, or none of those properties. We might want to count how many of those objects contain *at least* one property. The second equality in 3.2.3 could be used to prove the next statement.

Proposition 3.3.2. If $N_{i_1,i_2,\ldots,i_r}$ denotes the number of objects that contain properties $i_1, i_2, \ldots, i_r$ at the same time, then the number of those objects that have at least one property equals

$$\sum_{r=1}^{k}(-1)^{r-1} \sum_{1\leq i_1<i_2<\cdots<i_r\leq k} N_{i_1,i_2,\ldots,i_r}.$$

The formula might look complicated but it isn't, and what is more, it comes to use quite often.

3.3.5 Patterns and zeroes/ones

One very nice series of zeroes and ones that gives rise to a number of problems is the parity of the sum of the digits of n when written in its binary form. For $n = 0, 1, \ldots$ it looks like that:

$$0, 1|1, 0|1, 0, 0, 1|1, 0, 0, 1, 0, 1, 1, 0|1 \ldots$$

Did you catch the pattern? This seties appears unexpectedly in many situations.

Problem 3.3.23. (IMO SL, 1996) Let the sequence $a(n), n = 1, 2, 3, \ldots$ be generated as follows with $a(1) = 0$ and for $n > 1$:

$$a(n) = a\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + (-1)^{\frac{n(n+1)}{2}}.$$

1. Determine the maximum and minimum value of $a(n)$ over $n \leq 1996$ and find all $n \leq 1996$ for which these extreme values are attained.

2. How many terms $a(n)$, for $n \leq 1996$, are equal to 0?

Problem 3.3.24. (Leningrad Olympiad?) Given are 2^n sequences of 0s and 1s such that none of them is the beginning of another. Prove that their total length is at least $n2^n$.

Problem 3.3.25. Show that for any n there is a sequence A of $2^n + n - 1$ zeros and ones such that every sequence of n zeroes and ones appears in A.

Problem 3.3.26. (Bulgarian MO, 1999) In a competition, 8 judges marked each contestant with *yes* and *no*. For every two contestants, there were two judges marking both of them *yes*, two judges marking both of them *no*, two judges markinf the first one *yes* and the second one *no*, and two judges marking the first one *no* and the second one *yes*. What is the largest possible number of contestants?

Problem 3.3.27. Prove that for any n, the numbers $1, 2, \ldots, 3^n$ can be split into three groups, A, B, and C, so that

$$\sum_{a \in A} a^k = \sum_{b \in B} b^k = \sum_{c \in C} c^k$$

for $k = 0, 1, 2, \ldots, n - 1$.

3.4 Classes of equivalence

Consider a set of different states and a rule that allows us to jump from one state to another. For a given element, consider the set of all elements that are accessible from it. Also, suppose that such accessibility is mutual, that is, a is accessible from b whenever b is accessible from a. For example, the rule might allow us to "jump back" from b to a or it might allow us to complete a full cycle from a through b and back to a. In that case, all elements accessible from a given element form a *class of equivalence*: every two states within the class are accessible from one another, but not two elements from different classes are accessible from one another.

A typical example is a graph, where "a jump" is allowed from one vertex to another whenever the vertices are connected. In that case, every conected subgraph is a class of equivalence and the big graph can be partitioned into such classes. Another example would be dividing the set of integers into 7 classes of equivalence, where we are allowed to jump between any two numbers that differ by 7. It is apparent that each class of equivalence consists of numbers that give the same remainder modulo 7.

The concept is simple, but its applications are very powerful, especially in cases where the classes have the same number of elements. In such cases, we might be interested in finding a bijection between their elements.

3.4.1 Orbits

Suppose we are given a bijection f from a *finite* set A in itself. For every x, consider the series $x_0 = x, x_n = f(x_{n-1})$ for $n = 1, 2, \ldots$. This series must repeat itself at some point, since A is finite. Once it repeats itself for the first time, it forms an *orbit* of certain length. This orbit is a class of equivalence with respect to f: no two orbits intersect and no orbit "self intersects" (since the bijective f is also injective,) yet every two elements of an orbit are accessible from one another.

Problem 3.4.1. (IMO SL, 1996) let V be a finitive set and f and g be two bijective functions from to V to V. Define the sets T and S as follows:

$$S = \{w \in V : f(f(w)) = g(g(w))\}$$
$$T = \{w \in V : f(g(w)) = g(f(w))\}$$

If $S \cup T = V$, prove that for each $w \in V$, $f(w) \in S$ if and only if $g(w) \in S$.

3.4.2 Bijections

Proving that two sets have the same number of elements might not be possible directly by counting them separately, but by finding a bijection between them might save us.

Problem 3.4.2. Consider a coordinate system and all *integer* points in it(points with integer coordinates.) Prove that any two circles with diameters OA and OB such that $OA = OB$ pass through the same number of integer points.

Problem 3.4.3. (IMO SL, 2001) For a positive integer n define a sequence of zeros and ones to be balanced if it contains n zeros and n ones. Two balanced sequences a and b are calle *neighbors* if you can move one of the $2n$ symbols of a to another position to form b. For instance, when $n = 4$, the balanced sequences 01101001 and 00110101 are neighbors because the third (or fourth) zero in the first sequence can be moved to the first or second position to form the second sequence. Prove that there is a set S of at most $\frac{\binom{2n}{n}}{n+1}$ balanced sequences such that every balanced sequence is equal to or is a neighbor of at least one sequence in S.

Problem 3.4.4. (Russian MO, 1976) Given is a regular n-gon inscribed in a unit circle, at whose vertices either $+1$ or -1 is written. One is allowed to simlutaneously change the signs of those vertices that form a regular k-gon for some $k \geq 2$ (a two-gon simply means a diagonal through the center.) Prove that for all $n > 2$ there exists an initial configuration from which it is impossible to turn all -1s into $+1$s. In general, find the number of classes of equivalence.

Problem 3.4.5. There are n parking spots in a line, initially all empty, ordered from 1 to n from the entrance to the exit. Before one enters the lot, they pick a parking spot at random and go straight there. If it is occupied, they keep searching for the next available spot. If there are no such spots, they drive off. What is the probability that the first n to enter will find parking spots?

Problem 3.4.6. (IMO, 1995) Let p be an odd prime number. How many p-element subsets A of $\{1, 2, 3, \ldots 2p\}$ are there the sum of whose elements is divisible by p?

3.5 Invariants and the likes

A whole set of combinatorial problems sounds as follows: given is an initial state A and a rule for moving from one state to another, the goal bein to reach state B. Typical questions are

1. prove it is *not possible* to reach state B;
2. prove it is *possible* to reachc state B;
3. find the least number of moves needed to reach state B;
4. prove that, no matter what, you are going to end up with state B.

For the first set of questions, we want to find a function (ideally numeric) of that state which does not change from one move to the next. Such functions are called *invariant*, since, as the name suggests, something is not varying no matter how we move across states. To deal with such questions, all we need to do is find the invariant function and show that the values at state A and state B are different to show one can never reach state B.

For the second set of questions, we want to find a function that moves continuously over the real numbers or with ± 1 steps over integer numbers. In doing so, we want to show that we can get from state A to state U and then to state D such that $f(U) \geq f(B) \geq f(D)$. If we want to show the existence of state B for which $f(B)$ is within a certain range of length l, we just need to show that the step of f from one state to the next is no larger than l.

The third set of questions involves finding a function which does not change *too much* and use the fact that $f(A)$ and $f(B)$ are far apart.

The final set of questions involves finding a monotonous (possibly integer) function which reaches its minimum in a finite number of moves, and show that B is the only state with that property. Such functions are often dubbed *semi-invariants*, but we use the word more freely here to denote any function that vhanges in a predictable fashion.

Notice that many inequalities $f(x_1, x_2, \ldots, x_n) \geq 0$ lend themselves to similar techniques, e.g. manipulating the numbers until equality is reached and showing that the number on the right-hand sidehas been decreasing al the while. That is known as the method of Sturm, but that is not the focus of this chapter.

3.5.1 Invariants

The following example illustrates a typical use of an invariant.

Problem 3.5.1. (Kvant, 1985) On a given island, there are 45 chameleons: 13 red, 15 green, and 17 blue. Whenever two chameleons of different colors meet, they change their color to the third one (e.g. a red an a green turn both blue.) Can it happen that after some time all chameleons turn into one and the same color?

Problem 3.5.2. Given is a sequence of ordered couples $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$, such that $(a_1, b_1) = (0, 1)$ and for $i = 1, 2, \ldots, n-1$, (a_{i+1}, b_{i+1}) equals either

$$(-a_i, b_i) \text{ or } \left(\frac{5}{4}a_i + \frac{3}{4}b_i, \frac{5}{4}b_i + \frac{3}{4}a_i\right).$$

Prove that $\prod_i (a_i + b_i) \geq 1$.

Problem 3.5.3. There are $2n$ numbers $a_1, \frac{1}{a_1}, a_2, \frac{1}{a_2}, \ldots, a_n, \frac{1}{a_n}$ written on a board. One is permitted to pick any two numbers a and b whose product is not 1 from the board, erase them, and write the number $\frac{a+b}{ab-1}$. If there is one number standing at the end, what is it?

Problem 3.5.4. (IMO SL, 1994) Let $f(x) = \frac{x^2+1}{2x}$ for $x \neq 0$. Define $f^{(0)}(x) = x$ and $f^{(n)}(x) = f\left(f^{(n-1)}(x)\right)$ for all positive integers n and $x \neq 0$. Prove that for all non-negative integers n and $x \neq \{-1, 0, 1\}$

$$\frac{f^{(n)}(x)}{f^{(n+1)}(x)} = 1 + \frac{1}{f\left(\left(\frac{x+1}{x-1}\right)^{2n}\right)}.$$

Problem 3.5.5. (IMO, 1993)(*) On an infinite chessboard, a solitaire game is played as follows: at the start, we have n^2 pieces occupying a square of side n. The only allowed move is to jump over an occupied square to an unoccupied one, and the piece which has been jumped over is removed. For which n can the game end with only one piece remaining on the board?

3.5.2 Semi-invariants

Similarly to functions or properties which stay constant (invariant), we can use functions which change in only one direction, i.e. monotonous functions. That way we can prove it is impossible to reach state B from state A if the function starting at A will move away from its value at B with every move. On the other hand, if state B is reachable, we can use the fact that the function changes by "only so much" in order to prove that it takes *at least* so many moves to reach B from A.

Problem 3.5.6. The numbers $1, 2, \ldots, n$ are written in a sequence. In one move, one is permitted to switch the places of two adjacent numbers. What is the minimal number of moves required to reverse the order of the n numbers?

Problem 3.5.7. 10 people are sitting at a round table. There are some walnuts in front of each of them, 100 walnuts in all. After a certain signal, *at the same time*, every person passes some of his walnuts to the person sitting to his right:

if he has an even number of walnuts, he passes half of them; otherwise, he passes one walnut plus half of the remaining walnuts. This procedure is repeated over and over again. Prove that, eventually, everyone will have exactly 10 walnuts.

Problem 3.5.8. (IMO, 1986) To each vertex of a regular *pentagon* an integer is assigned, so that the sum of all five numbers is positive. If three consecutive vertices are assigned the numbers x, y, z respectively, and $y < 0$, then the following operation is allowed: x, y, z are replaced by $x + y, -y, z + y$ respectively. Such an operation is performed repeatedly as long as at least one of the five numbers is negative. Determine whether this procedure necessarily comes to an end after a finite number of steps.

Problem 3.5.9. (Singapore Training, 2009) Given a finite graph, prove that it is always possible to assign each vertex one of two colors so that every vertex is connected to at least as many vertices of the opposite color as vertices of its own color.

Problem 3.5.10. (Russian MO) Given is a deck of 33 cards. On each move, one is allowed to select a few of the cards and, *without changing their order*, pull them out and place them on top of the deck. What is the minimum number of moves that it takes to reverse the original order in the deck?

Problem 3.5.11. (Kvant, 1990) A table 50×50 is filled with $+1$s and -1s such that the sum of all numbers in the table does not exceed 100 in absolute magnitude. Prove that there exists a square 25×25 the sum of whose numbers does not exceed 25 in magnitude.

Problem 3.5.12. (Bulgarian MO, 2001) For a given sequence of numbers $a_1, a_2, \ldots, a_n$, on each move, one is allowed to switch two consecutive blocks of numbers in the sequence, i.e. from

$$a_1, a_2, \ldots, a_i, \underbrace{a_{i+1}, a_{i+2}, \ldots, a_j}_{A}, \underbrace{a_{j+1}, a_{j+2}, \ldots, a_k}_{B}, a_{k+1}, \ldots, a_n,$$

one can obtain by swapping the blocks A and B

$$a_1, a_2, \ldots, a_i, \underbrace{a_{j+1}, a_{j+2}, \ldots, a_k}_{B}, \underbrace{a_{i+1}, a_{i+2}, \ldots, a_j}_{A}, a_{k+1}, \ldots, a_n.$$

Find the minimum number of such moves required to change the sequence $1, 2, 3, \ldots, n$ to $n, n-1, n-2, \ldots, 1$.

Problem 3.5.13. A girl is having a birthday party and wants to cut the cake in advance so that, no matter whether p or q guests turn up, she be able to give each guest an equal amount of cake (potentially in several slices). What is the minimum number of slices she needs to cut the cake into?

Problem 3.5.14. (IMO SL, 1998) A rectangular array of numbers is given. In each row and each column, the sum of all numbers is an integer. Prove that each nonintegral number x in the array can be changed to either $\lfloor x \rfloor$ or $\lceil x \rceil$ so that the row-sums and column-sums remain unchanged.

Problem 3.5.15. (IMO, 2000) Let $n \geq 2$ be a positive integer and λ a positive real number. Initially there are n fleas on a horizontal line, not all at the same point. We define a move as choosing two fleas at some points A and B, with A to the left of B, and letting the flea from A jump over the flea from B to the point C so that $\frac{BC}{AB} = \lambda$.
Determine all values of λ such that, for any point M on the line and for any initial position of the n fleas, there exists a sequence of moves that will take them all to the right of M.

3.5.3 Parity

The most common invariant is parity. When some process preserves the parity of some function, this process lead to a state where the function is of the opposite parity. In case there is no process but rather a static configuration which we have to prove cannot exists, we usually to count the same onject in two different ways and prove that the supposed parities are different.

Here are a few examples.

Problem 3.5.16. (Moscow Olympiad) Can one number each of the $2n$ pins of a radiolapmp and the $2n$ slots of its circular socket with $1, 2, \ldots, 2n$ in such a way that no matter how the lamp is put into the socket at least one slot be matched against a pin with the same number?

Problem 3.5.17. (IMO SL, 1997) An $n \times n$ matrix with entries from $\{1, 2, \ldots, 2n-1\}$ is called *coveralls matrix* if for each $i \in [1, n]$ the union of the i-th row and the i-th column contains $2n - 1$ distinct entries. Show that:

(i) There exist no coverall matrices for $n = 1997$;

(ii) Coverall matrices exist for infinitely many values of n.

Problem 3.5.18. (Russian MO, 1977) A square 100×100 is divided into 100^2 unit squares. Several broken lines are drawn inside the square, their segments being sides of the little squares. Their endpointss lie on the border of the big square, but their segments do not. Moreover, the broken lines are neither pairwise- nor self-intersecting, i.e. no two lines have a common point. Prove that there exists a vertex of a little square, which is not a vertex of the big square and which does not belong to any broken line.

Problem 3.5.19. (IMO SL, 1998) A solitaire game is played on an $m \times n$ rectangular board, using mn markers which are white on one side and black on the other. Initially, each square of the board contains a marker with its white side up, except for one corner square, which contains a marker with its black side up. On each move, one may take away one marker with its black side up, but must then turn over all markers which are in squares having an edge in common with the square of the removed marker. Determine all pairs (m, n) of positive integers such that all markers can be removed from the board.

Problem 3.5.20. (Tournament of Towns, Spring 1999) A move of the rook consists of passing to a neighboring cell in either the horizontal, or the vertical direction. After 64 moves the rook visited all cells of the 8×8 chessboard and returned back to the initial cell. Prove that the number of moves in the vertical direction and the number of moves in the horizontal direction are distinct.

3.5.4 Continuity

Similar to semi-invariants, where we use monotonous functions, we can use functions that only change by a little bit to prove that they have values in a certain interval. We can use integer functions changing by no more than 1 to show that a certain value x is attained, so long as we can give examples of the function taking values above and below x.

Problem 3.5.21. (IMO, 1997) Let $x_1, x_2, \ldots, x_n$ be real numbers satisfying the conditions:

$$\left| \begin{array}{ccl} |x_1 + x_2 + \cdots + x_n| & = & 1 \\ |x_i| & \leq & \frac{n+1}{2} \qquad \text{for } i = 1, 2, \ldots, n. \end{array} \right.$$

Show that there exists a permutation $y_1, y_2, \ldots, y_n$ of $x_1, x_2, \ldots, x_n$ such that

$$|y_1 + 2y_2 + \cdots + ny_n| \leq \frac{n+1}{2}.$$

Problem 3.5.22. (Kvant, 1988) There are 19 positive integers, each no bigger than 88, written in the first line of a notebook; in the second line, there are 88 positive integers, each no bigger than 19. We shall call a *cut* a sequence of one or several numbers in a row in one of the lines. Prove that there exist two cuts in each line with the sums of their numbers equal.

Problem 3.5.23. (IMO SL, 1996) Let p, q, n be three positive integers with $p + q < n$. Let $(x_0, x_1, \ldots, x_n)$ be an $(n+1)$-tuple of integers satisfying the following two conditions:

(i) $x_0 = x_n = 0$;

(ii) For each i, $1 \leq i \leq n$, either $x_i - x_{i-1} = p$ or $x_i - x_{i-1} = -q$.

Show that there exist indices $i < j$ with $(i, j) \neq (0, n)$, such that $x_i = x_j$.

3.6 Strategies

Most mathematical problems describe a *fixed* process or situation and ask about its end result. Typical examples are all geometrical problems, functional equations, recurrent equations (sequences,) and counting problems. Those problems are fundamentally different from ones where we are given a *rule* instead of a fixed situation. With the risk of confusing you, I would like to throw in all inequalities in this category. The main difference is that this time we are given optionality.

If we want to prove some statement, we want to shoot for the most unpleasant case, in other words, we have to play the devil's advocate in approaching such problems. For example, in proving an inequality $f > g$, we might want to consider the difference $f - g$, find its minimum (the "worst" case,) and show that it is at least 0. Or in a game where two players take turns and we want to show that the first one can win no matter what, we must assume that the second player will be giving an arm and a leg not to lose.

Consider the following classic.

Problem 3.6.1. Two people take turns in taking toothpicks from three piles which originally contain 3,5, and 8 toothpicks respectively. On each turn, the current player might take as many toothpicks as he wants but only from one pile. The winner is the one who picks the last toothpicks. Show that the player who plays first has a winning strategy, that is, he can always win no matter how his/her opponent plays.

Describing a winning strategy means describing a rule that guarantees a win.

3.6.1 Exercises

Problem 3.6.2. (Bulgarian MO, 1995) Two players take turns in taking stones form a heap that initially contains n stones. On every turn, the player who takes stones must take at least one and not all stones. The winner is the one who takes the last stone. For each n, determine which player has a winning strategy.

Problem 3.6.3. The two sides of each of $2n$ cards are labeled with numbers from 1 to $2n$, so that each number between 1 and $2n$ appears on two sides (possibly of the same card.) Only one side of each card is visible to us at a time. In one turn, we are allowed to flip a card, and the final goal is to end up with $2n$ different numbers facing up.

- Prove that $3n - 2$ moves are always sufficient;
- Prove that $3n - 3$ moves are not always sufficient.

Problem 3.6.4. (IMO SL, 1996) A finite number of coins are placed on an infinite row of squares. A sequence of moves is performed as follows: at each stage a square containing more than one coin is chosen. Two coins are taken from this square; one of them is placed on the square immediately to the left while the other is placed on the square immediately to the right of the chosen square. The sequence terminates if at some point there is at most one coin on each square. Given some initial configuration, show that any legal sequence of moves will terminate after the same number of steps and with the same final configuration.

3.7 Principle of the extreme element

3.7.1 Exercises

Problem 3.7.1. (Putnam) An integer is written on each of the n beads of a necklace so that the sum of all integers is positive. Prove that the necklace can be cut in some place and the beads numbered consecutively with $1, 2, \dots, n$ in such a way that for each $k = 1, 2, \dots, n$ the sum of the integers written on the first k beads be non-negative.

Problem 3.7.2. (Moskow Olympiad?) 101 numbers are written on a board in a series. Prove that one can erase 90 of them so that the remaining eleven form either an increasing or a decreasing sequence.

3.8 Graphs

Non-technically speaking, a graph is a collection of *vertices* (points) and *edges* (lines) between some of the points. When we have some objects and want to express some relationship between them, this can often be illustrated as a graph whose vertices represent the objects and whose edges represent relationships between those objects.
For example, if we have a party where every two people either know or do not know each other, we can represent this relationship as a graph with a vertex for every person and an edge for every pair of people who know each other.
To be able to tell when we need to introduce a graph and do more than just stare at it, we need some very basic graph theory: a couple of definitions and simple properties.

Definition 3.8.1. (Connected graph) A graph is called *connected* when there exists a path between every two vertices A and B, that is, a sequence $A_0 = A, A_1, A_2, \dots, A_n = B$ such that $A_{i-1}A_i$ is an edge of the graph for $i = 1, 2, \dots, n$.

Proposition 3.8.1. Every finite graph is a collection of connected graphs that do not share any common vertices and edges.

Definition 3.8.2. (Trees) A *cycle* is a path that starts and terminates at the same point. A *connected* graph that does not contain any cycles is called *tree*.

Proposition 3.8.2. A tree with n vertices contains $n - 1$ edges.

Definition 3.8.3. (Euler Graphs) A graph all of whose vertices are of degree is called *Euler graph*.

Proposition 3.8.3. Every Euler graph contains a cycle that includes every edge exactly once.

Definition 3.8.4. (Bipartite graphs) A graph that could have its vertices divided into two groups, A and B, with no edges with both vertices in A or in B is called *bipartite*

Proposition 3.8.4. A graph is bipartite iff it has no cycles of even length.

3.8.1 Hall's theorem

Suppose we are given a collection of sets and want to choose one element from each set so that they are all distinct. This is called a *system of distinct representatives* (or SDR) of the collection. Examples of why in our right minds we would need that is given in the exercies. But before we go to them, we shall prove a fundamental fact about SDR.

Proposition 3.8.5. Let $\mathcal{A} = \{A_1, A_2, \dots, A_n\}$ be a collection of n finite sets. If for any $k \in [1, n]$ and $1 \le i_1 < i_2 < \cdots < i_k \le n$ holds

$$|A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_k}| \ge k,$$

then there exists a SDR for $\mathcal{A}$.

3.8.2 Exercises

Problem 3.8.1. Given are several *different* sequences containing n ones and m zeroes each for some $n > m$. Prove that in each sequence, exactly one 1 could be changed to 0 so that the new sequences with $n-1$ ones and $m+1$ zeroes are again pairwise different.

Problem 3.8.2. In some city, there are 50 school principals and 50 deputies. Each principal has a direct telephone line with exactly k deputies and each deputy has a direct line with exactly k principals. Prove that principals and deputies can be paired up so that in each pair the principal and the deputy have a direct line between them.

Problem 3.8.3. Given are k different colors and n balls of each color. They are thrown in k boxes, n balls per box. Show that one can select one ball from each box so that all selected balls have different collors.

Problem 3.8.4. (Singapore Training, 2009) In a tournament with $n \ge 2$ players, every two players meet exactly once, resulting in a win for one of them. A player, say player A, is awarded a Grand Prize if for any other player B, either A beat B or there was another player C such that A beat C and C beat B. Show that

(i) the tournament must have at least one Grand Prize winner;

(ii) if there is exactly one such winner, then he beat everyone else.

Problem 3.8.5. (Kvant, 1988) There are 21 cities in a country, and for each two of them there is at least one airline company that provides a direct flight between them. Each airline serves exactly 5 cities and provides direct flights between every two of them. What is the minimal number of airline companies possible?

Problem 3.8.6. (Spring Competition, Bulgaria, 1995) Given are $n > 4$ points in the plane, no three of them collinear. Prove that among any $n+1$ triangles with vertices those points, there are two triangles that share exactly one vertex.

Problem 3.8.7. (Russian MO, 1977) There are direct flights between every two cities in some country, and the prices are known in advance. Two guys – not necessarily starting from the same city – want to travel through all the cities using only direct flights. Among the cities he hasn't visited, Person A always picks the one that offers the cheapest rate (if there are several, he picks one at random) until he reaches the last city. Person B does the same, except he always goes for the most expensive destination. Prove that, in the end, person A will have spend no more than person B for travel.

Problem 3.8.8. (Bulgarian MO, 2001) Find the least positive integer n for which there exists a group of n people such that:

(i) There is no group of four every two of which are friends;

(ii) For any choice of $k \geq 1$ people among which there are no friends, there exists a group of three among the rest $n-k$ every two of whom are friends.

Problem 3.8.9. (Koenig's theorem) Several checkers have been placed in the squares of an $m \times n$ chessboard. A *covering* is a selection of rows and columns such that every checker lies in at least one selected row or column. Prove that the *maximum* number of checkers no two of which lie in the same row or column is the same as the *minimum* number of rows and columns that provide a covering.

Problem 3.8.10. (Moskow Olympiad) King Arthur is trying to sit $2n$ knights around a round table so that no one of them sits next to an enemy of his. Prove that this is always posible to do if it is known that every knight has no more than $n-1$ enemies.

Problem 3.8.11. (Russian Olympiad, 2001) There are 2001 towns in a country. For any town, there exists a road going out of it, and there is no town directly connected to all other towns. A set of towns D is called *dominant* if any town that does not belong to D is directly connected by a road with some town from D. It is known that no dominant set contains fewer than k towns. Prove that the country can be split into $2001-k$ republics such that no two towns from the same republic be connected by a road.

Problem 3.8.12. (IMO, 2007) At a mathematical competition, some competitors are friends. Friendship is always mutual. Call a group of competitors a *clique* if every two in the group are friends (in particular, any group of fewer than two competitiors is a clique.) The number of members of a clique is called its size.
Given that, in this competition, the largest size of a clique is even, prove that the competitors can be arranged into two rooms such that the largest size of a clique contained in one room is the same as the largest size of a clique contained in the other room.

3.9 Combinatoric geometry

This class encompasses problems that look like geometry but are indeed pure combinatorics. Geometry is only auxiliary in presenting a combinatorial situation, and the tools we use are of the sort of coloring, invariants, and counting. Here are some possibly useful facts.

Theorem 3.9.1. *(Helly's theorem) Given n convex figures in the plane, if every three figures share a point, then all n share a point.*

Theorem 3.9.2. *(Euler's formula) For any* convex *polyhedron holds*

$$V - E + F = 2,$$

where V is the number of vertices, E is the number of edges, and F is the number of faces.

3.9.1 Averaging

Suppose a segment AB of length l is projected onto a line with which it closes an angle ϕ. The length of the projection is obviously $l|\cos(\phi)|$. If we "average" over all possible ϕ, that is, integrate the function $l|\cos(\phi)|$ over $[0, \pi]$, we get that the average length of the projection is $\frac{l}{\pi}$. That means that there exist lines such that the projections of AB onto them have lengths l_1 and l_2 respectively, such that $l_1 \leq \frac{l}{\pi} \leq l_2$.

That might not appear like a striking revelation, as the projections of AB onto the lines perpendicular and parallel to it have lenghts 0 and l respectively. The observation gains much more importance when we have many segments which are not necessarily aligned.

3.9.2 Coloring

In a sense, coloring is a generic term that relates to binning our objects into groups, e.g. by "painting" them in black or white. It is closely related to the invariants technique that we described. For example, if we have a board game, we might want to paint the squares of the board in two colors so that the numbers of checkers on white and on black squares are of the same parity.

3.9.3 Exercises

Problem 3.9.1. (Kvant, 1973) Several chords are drawn in a circle of radius 1 so that no diameter crosses more than k of them. Prove that the sum of the lengths of all chords is *less* than πk.

Problem 3.9.2. (Kvant, 1992) Every face of a convex polyhedron is a polygon with an even number of sides. Is it always possible to color the edges in two colors so that each face have an equal number of edges of either color?

Problem 3.9.3. (IMO, 1986) Given a finite set of points in the plane, each with integer coordinates, is it always possible to color the points red or white so that for any straight line l parallel to one of the coordinate axes the difference (in absolute value) between the numbers of white and red points on l is not greater than 1?

Problem 3.9.4. A finite number of the squares of an infinite chessboard are painted in red, and the rest are painted in blue. There is a line that *crosses* exactly 2010 red squares. Prove that for every $k = 0, 1, 2, \ldots, 2010$ there exists a line that crosses exactly k red squares. (A line is said to be crossing a square when it passes through its interior and not just through the sides or the vertices).

Problem 3.9.5. (IMO, 1965) Prove that the diameter of a set of $n > 3$ points in the plane (a segment of maximal length between any two of those points) appears no more than $n - 1$ times.

Problem 3.9.6. A square $n \times n$ is split into unit squares. Several lines are drawn so that every unit square shares an internal point with at least one of them. Prove that their number is larger than $\left(2 - \sqrt{2}\right) n$.

Problem 3.9.7. A *parabola* in the plane shall be called the set of all points *on or above* the curve $y = ax^2$ for some $a > 0$ and some coordinate system in the plane. Is it possible to cover the whole plane with a finite number of parabolas?

Problem 3.9.8. A rectangle is called *semi-integer* if at least one of its sides is integer. Prove that if a rectangle is cut into semi-integer rectangles, then the rectangle itself is semi-integer.

Problem 3.9.9. (Tournament of towns, 1989) Given are n lines in the plane, no two of which are parallel and no two of which pass through the same point. Prove that in every region they divide the plane, one can write an integer number different from 0 and not exceeding n in absolute value so that for every line the sums of the numbers of either side be equal to 0.

Problem 3.9.10. Given is a coordinate system and a figure of area $S > 1$. Prove that there exist two points in this figure $A(x_1, y_1)$ and $B(x_2, y_2)$ such that $x_1 - x_2$ and $y_1 - y_2$ are both integers.

Problem 3.9.11. Given a sphere of radius 1 and a curve of length $l < 2\pi$ on it, prove that there exists a hemisphere that contains the curve.

Problem 3.9.12. A rectangle is cut into squares. Prove that it can be cut into equal squares.

Problem 3.9.13. (Kvant 1980) Given are several squares with total area 4. Prove that one can cover a quare of area 1 with them.

Problem 3.9.14. (Tournament of Towns, 1989)

(i) There are $3n$ asteriks in the fields of a $2n \times 2n$ table. Prove that one can erase n rows and n columns so that all asteriks be erased from the table.

(ii) Prove that one can put $3n + 1$ asteriks in the fields a $2n \times 2n$ table so that after erasing any n rows and any n columns there still be at least one asterik left.

Problem 3.9.15. Prove that any convex polygon with circumference l can be projected onto a line so that the projection has length at least $\frac{l}{\pi}$

Problem 3.9.16. Given are $2n$ red and $2n$ blue points in the plane, no three of them collinear. Prove that there exists a line such that there are n red and n blue points on either side of it.

Problem 3.9.17. (Kvant, 1982) Is it possible to cover a circle of diameter 1 with several stripes the sum of whose widths is strictly less than 1?

Problem 3.9.18. (prob:vlad:cross) A *cross* a figure consisting of two perpendicular mutually bisecting segments of length 2. Prove that if 12650 such crosses are fully placed inside a 100×100 square, then two of them must intersect. Is there a better bound than 12650?

Problem 3.9.19. (Tournament of Towns?) Two players take turns in putting "X" (for player one) and "O" (for player two) on an infinite chessboard. The goal of each player is to place 11 consecutive of his marks in a row, vertical, or diagonal. Prove that no player has a winning strategy, that is, player two can always prevent player one from winning and vice versa.

Problem 3.9.20. (Tournament of Towns, Autumn 1997) Every side of an equilateral triangle is divided into n equal segments, and through the points of division are drawn lines parallel to the sides of the triangle. Thus the original triangle is split into n^2 small triangles. What is the maximum number of small triangles that can be market so that no two of them be situated between two adjacent parallel lines if

(i) $n = 10$;

(ii) $n = 9$?

(iii) The same question for an arbitrary n.

Problem 3.9.21. Given is the lattice $\mathbb{Z}_0^2$ with 0 written in $(0, 0)$. On each move, one is allowed to write a number in the empty (x, y) so long as all other cells (x_1, y_1) with $x_1 \leq x$ and $y_1 \leq y$ have been filled, and the number must be the smallest non-negative integer that does not appear in the same row or column. For any m and n, what is the number that sits in m, n?

Problem 3.9.22. (IMO, 1999) Let n be an *even* positive integer. We say that two different cells of a board are *neighboring* if they have a common side. Find the minimal number of cells on the $n \times n$ board that must be marked so that any cell (marked or not marked) has a marked neighboring cell.

Problem 3.9.23. (Kvant, 1970) The sides of a equilateral triangle are divided into n equal segments. Lines parallel to the sides of the triangle through those points split it into n^2 triangles. We call a *chain* a sequence of triangles which does not contain any triangle twice and where every triangle shares a side with the previous one. What is the maximal number of triangles in a chain?

3.10 Generating functions

The intersection of combinatorics and algebra happens when we attach combinatorial meaning to the coefficients of a polynomial. This polynomial is called a *generating function* of the combinatorial object at hand. We have already seen examples of generating functions when we defined the binomial coefficients. Using $(1+x)^n = (1+x)(1+x)^{n-1}$ we proved 3.2.1, that is, using algebraic properties of the generating function we proved a combninatorial equation.
Another example are Catalan's numbers (see 1.3.6). If we define $f(x) = C_0 + C_1x + C_2x^2 + \cdots$ and the definition of catalan's numbers, it is not hard to show that

$$f(x) = 1 + xf^2(x).$$

Useless at first sight, it implies that (after rejecting one of the roots of the quadratic equation)

$$f(x) = \frac{\sqrt{1-4x}+1}{2}$$

and from

$$\sqrt{1-4x} = (1-4x)^{\frac{1}{2}} = \sum_{n\geq 0} \binom{\frac{1}{2}}{n} (-4)^n x^n$$

we easily get $C_n = \dfrac{\binom{2n}{n}}{n+1}$.

3.10.1 Complex numbers

Using properties of the roots of unity, that is $\omega_k = e^{\frac{2i\pi k}{n}}$, either as roots of the polynomial $x^n - 1$ or as a cyclic group, we can prove some pretty interesting combinatorial or geometrical facts.

Problem 3.10.1. (IMO, 1990) Prove that there exists a convex 1990-gon with the following two properties :

(i) All angles are equal;

(ii) The lengths of the 1990 sides are the numbers $1^2, 2^2, 3^2, \ldots, 1990^2$ in some order.

Problem 3.10.2. (Russian Olympiad?) The vertices of a regular n-gon are painted in different colors so that all vertices of the same color form a regular polygon. Prove that two of those polygons have the same number of vertices.

3.10.2 Exercises

Problem 3.10.3. A table 6×21 is filled with the numbers $0, 1, \ldots, 6$ so that every numbers appears 18 times. It is known that for every four numbers in the vertices of a rectangle with sides parallel to the table, the sums of the numbers at the ends of both diagonals are congruent modulo 7. Prove that each row contains 3 zeroes, 3 ones, 3 twos etc.

Problem 3.10.4. (Bulgarian IMO selection, 2001) For an arbitrary set $S\{a_1, a_2, \ldots, a_k\}$ of integers with $1 \leq a_1 < a_2 < \cdots < a_k \leq 2000$ define the set

$$\Phi(S) = \begin{cases} \{a_1+1, a_2+1, \ldots, a_k+1\}, & \text{if } a_k < 2000; \\ \{1, 2, \ldots, 2000\} \backslash \{a_1+1, a_2+1, \ldots, a_{k-1}+1\}, & \text{if } a_k = 2000. \end{cases}$$

Prove that $\Phi^{(2001)}(S) = S$, where $\Phi^{(n)}(S)$ denotes the n-nth iteration of Φ.

Problem 3.10.5. Given are two *different* multisets $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$. (A multiset is a set, where the same element could be repeat, e.g. $\{1, 2, 3, 3\}$.) Prove that if the multisets $\{a_i + a_j | 1 \leq i < j \leq n\}$ and $\{b_i + b_j | 1 \leq i < j \leq n\}$ coincide, then n is a power of 2.

Problem 3.10.6. In how many ways can two sets A and B of non-negative integers be chosen so that

(i) Every number between 0 and $2^n - 1$ can be uniquely represented as the sum of an element from A and an element from B.

(ii) For any $a \in A$ and $b \in B$, $a + b < 2^n$.

Problem 3.10.7. (USAMO, 1999) Let p be a prime and let a, b, c, d be integers not divisible by p, such that

$$\left\{\frac{ra}{p}\right\} + \left\{\frac{rb}{p}\right\} + \left\{\frac{rc}{p}\right\} + \left\{\frac{rd}{p}\right\} = 2$$

for any integer r not divisible by p. Prove that p divides $(a+b)(a+c)(b+c)$. d

Problem 3.10.8. (IMO SL, 1998) Let $a_0, a_1, a_2, \ldots$ be an increasing sequence of non-negative integers such that every non-negative integer can be expressed uniquely in the form $a_i + 2a_j + 4a_k$, where i, j and k are not necessarily distinct. Determine a_{1998}.

Problem 3.10.9. (IMO, 2008) Let n and k be positive integers of the same parity with $k \geq n$. Let $2n$ lamps labelled $1, 2, \ldots, 2n$ be given, each of which can either be ON or OFF. Initially, all the lamps are OFF. We consider sequences of steps: at each step one of the lamps is switched (from ON to OFF or vice versa.)

- Let N be the number of such sequences consisting of k steps and resulting in the state where lamps $1, 2, \ldots, n$ are all ON, and the rest are all OFF;

- Let M be the number of such sequences consisting of k steps and resulting in the state where lamps $1, 2, \ldots, n$ are all ON, the rest are all OFF, and where none of the lamps $n+1$ through $2n$ is ever switched on.

Determine $\dfrac{N}{M}$.

Problem 3.10.10. (IMO SL, 1999) Suppose that every integer has been given one of the colors red, blue, green or yellow. Let x and y be odd integers so that $|x| \neq |y|$. Show that there are two integers of the same color whose difference has one of the following values: $x, y, x+y$ or $x-y$.

Problem 3.10.11. (IMO SL, 1999) Let p be a prime number. For each nonempty subset T of $\{0, 1, 2, 3, \ldots, p-1\}$, let $E(T)$ be the set of all $(p-1)$-tuples $(x_1, x_2, \ldots, x_{p-1})$, where each $x_i \in T$ and $x_1 + 2x_2 + \ldots + (p-1)x_{p-1}$ is divisible by p. If $|E(T)|$ denotes the number of elements of $E(T)$, prove that

$$|E(\{0,1,3\})| \geq |E(\{0,1,2\})|.$$

Chapter 4

Number Theory

Everything that has to do with integers and is not counting is thrown in here.

4.1 Basics

Every *natural* (positive integer number) n can be written in a unique way in the form

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$$

for some prime $p_1, p_2, \ldots, p_k$ and natural $\alpha_1, \alpha_2, \ldots, \alpha_k$. This form is called *canonical representation* of n.

Definition 4.1.1. For any positive integers n and k, we have $n = ak + b$ for some integers $a \geq 0$ and $k > b \geq 0$. The number a is the whole part of the division of n with k and is denoted by $\left\lfloor \frac{n}{k} \right\rfloor$, and b is the *remainder* of the division, i.e. the only number between 0 and $k-1$ for which $n \equiv b \pmod{k}$.

The symbols $\lceil x \rceil$ is also sometimes, yet rarely, used to denote the least integer bigger than or equal to x.

Proposition 4.1.1. If n, a, and b are positive integers, then

$$\left\lfloor \frac{\left\lfloor \frac{n}{a} \right\rfloor}{b} \right\rfloor = \left\lfloor \frac{n}{ab} \right\rfloor.$$

A very important property, which might seem ridiculous at first, but which proves indispensable over and over again is:

Proposition 4.1.2. If a and b are integers, then

$$a > b \Rightarrow a \geq b + 1.$$

An large number of problems rely on this subtle fact that whenever you have *strict* inequalities involving integers, you should always look to steal this "1" from somewhere. If the inequality is not strict, consider the case of equality separately and then focus on the "strict" part.

4.1.1 Exercises

Problem 4.1.1. (IMO SL, 1995) Let p be an odd prime. Determine positive integers x and y for which $x \leq y$ and $\sqrt{2p} - \sqrt{x} - \sqrt{y}$ is non-negative and as small as possible.

Problem 4.1.2. (IMO SL, 1999) Denote by S the set of all primes such the decimal representation of $\frac{1}{p}$ has the fundamental period divisible by 3. For every $p \in S$ such that $\frac{1}{p}$ has the fundamental period $3r$, one may write

$$\frac{1}{p} = 0, a_1 a_2 \dots a_{3r} a_1 a_2 \dots a_{3r} \dots,$$

where $r = r(p)$; for all integer $k \geq 1$ define

$$f(k,p) = a_k + a_{k+r(p)} + a_{k+2.r(p)}.$$

(i) Prove that S is infinite.

(ii) Find the highest value of $f(k,p)$ over $k \geq 1$ and $p \in S$.

Problem 4.1.3. Let p be a prime number and a and b be positive integers less than p. Prove that the equation $ax + by = p$ does have a solution in positive integers if and only if the sets $\left\{ \left\lfloor \frac{p}{a} \right\rfloor, \left\lfloor \frac{2p}{a} \right\rfloor \dots \left\lfloor \frac{(a-1)p}{a} \right\rfloor \right\}$ and $\left\{ \left\lfloor \frac{p}{b} \right\rfloor, \left\lfloor \frac{2p}{b} \right\rfloor, \dots, \left\lfloor \frac{(b-1)p}{b} \right\rfloor \right\}$ do not intersect.

Problem 4.1.4. (*) For a given prime $p > 2$, denote by $\underline{x}$ the remainder of x modulo p. For any integer a, define the function $f_a(m) = m + \underline{ma}$ for all integer m. Prove that there exist exactly $l2\sqrt{p}\rfloor$ integers $a \geq 0$ such that

$$\min_{m>0} f_a(m) > a.$$

Problem 4.1.5. Prove that for every positive integer n holds

$$\left| \left\{ \frac{n}{1} \right\} - \left\{ \frac{n}{2} \right\} + \left\{ \frac{n}{3} \right\} - \dots + (-1)^n \left\{ \frac{n}{n} \right\} \right| < \sqrt{2n}.$$

4.2 Diophantine equations

In general, solving an system of equations requires at least as many equations as there are unknown variables. However, with the added condition that the unknown variables be integer or positive integer, the set of solutions becomes manageable even with fewer equations. For example, the equation $6x^2+5y^2 = 74$ has infinitely many solutions in real numbers, (which form an allipse,) yet the only integer solutions are $(x, y) = (\pm 3, \pm 1)$.

There are multiple ways to approach such an equation and prove that it has no solutions other than the few obvious ones you have already found.

There are a number of approaches:

- Use a module to rule out the possibility of integer solutions; For example, $x^2 \equiv k \pmod 4$ has only solutions for $k = 0, 1$ and $x^3 \equiv k \pmod 9$ has only solutions for $k = 0, 1, 8$;
- When $a|bc$ for some co-prime b and c, we have $a = b_1c_1$ where $(b_1, c_1) = 1$ and $b_1|b$ and $c_1|c$.
- Use the fact that $x > y$ implies $x^2 > y^2 + 2y$ for *positive* integers x and y. In other words, once two different numbers are raised to some power, they become distant.
- Use the fact that $p|x^2 + y^2$ implies $p|x$ and $p|y$ for p-prome of the form $4k + 3$;
- Assume there is a solution and construct a "smaller" solution from it. Keep descending until a minimal solution is found, from where the series of solutions can be generated backward, or until it is clear that no minimal solution could possibly exist, in which case the original equation has no integral solution.

4.2.1 Pythagorean triangles

Suppose we want to find all rectangular triangles with integer sides, that is, solve the $x^2+y^2 = z^2$ in integers. Such equations form a small class of problems that can be approached in more or less the same way.

Proposition 4.2.1. All integer solutions of $x^2 + y^2 = z^2$ are given by

$$(x, y, z) = \left(kab, k\frac{a^2-b^2}{2}, k\frac{a^2+b^2}{2}\right)$$

for any two odd co-prime integers a and b and any integer k (assuming the places of x and y are interchangeable.)

4.2.2 Exercises

Problem 4.2.1. Solve the following equations in integers:

- $xyz = x + y + z$;
- $xyz = x + y + z + 1$;
- $\frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y} = 3$;
- $x^3 + y^3 + z^3 = 5$;
- $x^4 + y^4 - 2z^4 = 3$.

Problem 4.2.2. Prove that the equation $x^4+y^4 = z^2$ has no integer solutions.

Problem 4.2.3. (Singapore Training, 2009) Find all integer solutions of the following equation:

$$(x + y + z)^3 = 9(x^2y + y^2z + z^2x).$$

4.3 Euler's function and Fermat's little theorem

Definition 4.3.1. For everey natural n, define $\phi(n)$ as the number of integers between 1 and n which are co-prime with n. This is called Euler's function.

Proposition 4.3.1. Straight from the definition, $\phi(1) = 1$ and $\phi(p) = p-1$ for any prime p.

Proposition 4.3.2. If $(m, n) = 1$, then $\phi(mn) = \phi(m)\phi(n)$.

Proposition 4.3.3. If $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ is the canonical representation of n, then $\phi(n) = p_1^{a_1-1} p_2^{a_2-1} \cdots p_k^{a_k-1}(p_1 - 1)(p_2 - 1) \cdots (p_k - 1)$. *To prove, use propositions 4.3.1 and 4.3.2.*

Proposition 4.3.4. For any two co-prime integers a and n, $a^{\phi(n)} \equiv 1 \pmod{n}$. *To prove, use proposition 4.3.3 and induction on the power of the prime number.*

Proposition 4.3.5. $\sum_{d|n} \phi(d) = n$.

Proposition 4.3.6. $\sum_{(k,n)=1} k = \frac{n\phi(n)}{2}$.

Another symbol that is sometimes used alongside with ϕ is the Mobius function μ.

Definition 4.3.2. The Mobius function is defined as follows:

$$\mu(n) = \begin{cases} 0, & \text{if } n \text{ is divisible by a square;} \\ 1, & \text{if } n = 1; \\ (-1)^k, & \text{if } n \text{ is the product of } k \text{ distinct prime numbers.} \end{cases}$$

Like ϕ, the Mobius function possesses some similar properties.

Proposition 4.3.7. For any positive integer $k > 1$ holds

$$\sum_{k|d} \mu(k) \equiv 0 \pmod{p}.$$

Proposition 4.3.8. (Mobius inversion formula) If $f, g : \mathbb{N} \to \mathbb{N}$ satisfy

$$g(n) = \sum_{d|n} f(d)$$

for all $n \geq 1$, then for any $n \geq 1$ holds

$$f(n) = \sum_{d|n} \mu(d) g\left(\frac{n}{d}\right).$$

4.3.1 Quadratic residuals

For any prime $p > 2$, the numbers $1^2, 2^2, 3^2, \ldots, (p-1)^2$ give $\dfrac{p-1}{2}$ different remainders moduo p, since the ones in the first half are all different and the second half repeats the first half (notice that $x^2 \equiv (p-x)^2 \pmod{p}$.) These remainders, including $0 = 0^2$, are called *quadratic residuals* and the rest are called *quadratic non-residuals*.

Proposition 4.3.9. If p is a prime of the form $4k+3$, then $-x^2$ is a quadratic non-residual unless $x = 0$.

Proposition 4.3.10. For any prime p, the product of

- two quadratic non-residuals is a quadratic residual;
- of a residual and a non-residual is a quadratic non-residual.

Proposition 4.3.11. If $p > 3$ is prime, then the sum of all quadratic residuals modulo p is 0.

4.3.2 Primitive roots

Given a number n and an integer a co-prime with n, we know that there exists a number k such that $a^k - 1 \equiv 0 \pmod{n}$, for example $k = \phi(n)$. However, it is often useful to work with the *smallest* such number k, which is also called *index* of a modulo n.

Proposition 4.3.12. For every co-prime integers n and a, the index k of a modulo n divides any $l > 0$ such that $a^l \equiv 1 \pmod{n}$, and in particular, k divides $\phi(n)$.

Proposition 4.3.13. (Existence of primitive roots) For every p, there exists at least one number whose index is $p-1$. In fact, there are $\phi(p-1)$ such numbers. They are called *primitive roots*.

Proposition 4.3.14. Prove that for any prime p, the following identity holds in $\mathbb{Z}_p$:

$$(x-1)(x-2)(x-3)\cdots(x-(p-1)) = x^{p-1} - 1.$$

As a result, $(p-1)! \equiv -1 \pmod{p}$.

Proposition 4.3.15. For any prime p, the sum of its primitive roots is congruent $\mu(p-1)$.

4.3.3 Exercises

Problem 4.3.1. (Bulgarian IMO selection, 2007) Let $p = 4k+3$ be a prime number. Find the number of different residues modulo p of $(x^2+y^2)^2$ where $(x,p) = (y,p) = 1$.

Problem 4.3.2. (Singapore Training, 2009) Determine all pairs (p, n) of positive integers such that p is prime, $n \leq 2p$, and $(p-1)^n + 1$ is divisible by n^{p-1}.

Problem 4.3.3. (Bulgarian MO, 2000) Let $p \geq 3$ be a prime and $a_1, a_2, \ldots, a_{p-2}$ be a sequence of natural numbers such that p does not divide both a_k and $a_k^k - 1$ for $k = 1, 2, \ldots, p-2$. Prove that the product of some elements of the sequence equals 2 modulo p.

Problem 4.3.4. (Winter Competition, Bulgaria, 1997) Find all integers $m, n \geq 2$ such that

$$\frac{1 + m^{3^n} + m^{2 \cdot 3^n}}{n}$$

is integer.

Problem 4.3.5. (*) Let $p > 2$ be prime. Prove that

$$\sum_{i=1}^{p-1} 2^i i^{p-2} \equiv \sum_{i=1}^{\frac{p-1}{2}} i^{p-2}.$$

Problem 4.3.6. Prove that we can find an infinite set of positive integers of the form $2^n - 3$ (where n is a positive integer) every pair of which is relatively prime.

Problem 4.3.7. (IMO, 1990) Determine all integers $n > 1$ such that

$$\frac{2^n + 1}{n^2}$$

is an integer.

Problem 4.3.8. Given are positive integers a and n and a sequence $a_0, a_1, \ldots$ defined as

- $a_0 = a$;
- $a_n = a^{a_{n-1}}$ for $n = 1, 2, \ldots$

Prove that for any m such that $(a, m) = 1$ there exist numbers r and N such that $a_n = r \pmod{m}$ for all $n \geq N$.

Problem 4.3.9. (USAMO) If p is prime and $d \in \mathbb{N}$, then for any integer n there exist integers $x_1, x_2, \ldots, x_d$ such that p divides $x_1^d + x_2^d + \cdots + x_d^d - n$.

Problem 4.3.10. At the beginning, there are 1 and 1 written on a board. On every turn, one goes and writes between every two adjacent numbers their sum. For example, after the first move there will be $1, 2, 1$ on the board; after the second one there will be $1, 3, 2, 3, 1$ etc. Prove that, eventually, every integer n will appear exactly $\phi(n)$ times.

4.4 Gaussian Numbers

Just like extending $\mathbb{R}$ to $\mathbb{C}$ empowers us to solve problems that don't apriori mention complex numbers (e.g. finding a root of a polynomial with real coefficients,) extending the integers $\mathbb{Z}$ to a ring containing i helps us solve problems about integers. Let's start with some definitions.

Definition 4.4.1. A complex number with integer real and imaginary parts is called *Gaussian integer*. The set of such numbers is denoted as $\mathbb{Z}[i]$. The *norm* of a gaussian $a+bi$ is the number a^2+b^2, i.e. the square of its module, and the four numbers $\pm 1, \pm i$ with norm 1 are called *units*. For a given Gaussian integer z, the numbers $\pm z, \pm iz$ are called its *associates*.
Gaussian integers that cannot be represented as the product of two other gaussian integers with norm bigger than one are called *Gaussian primes*.

Proposition 4.4.1. Every prime p of the form $4k+3$ is also a Gaussian prime.

Proposition 4.4.2. Every prime p of the form $4k+1$ as well as $p=2$ equals the norm of some Gaussian prime.

Proposition 4.4.3. Every Gaussian prime is either the associate of a prime p of the form $4k+3$ or a solution of $z\bar{z}=p$ for some prime p of the form $4k+1$ or $p=2$.

Proposition 4.4.4. (Unique factorization) Every Gaussian integer z can be represented in the form

$$z = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$$

for some positive integers $\alpha_1, \alpha_2, \dots, \alpha_k$ and Gaussian primes $p_1, p_2, \dots, p_k$ in a unique way up to substition of some of the p_ls by their associates.

4.4.1 Exercises

Problem 4.4.1. (Bulgarian BO selection) For a given *even* integer n, let $M(n)$ denote the number of solutions of $x^2+y^2=n^2+1$ in positive integers and $N(n)$ be the number of divisor of n^2+1. Prove that $M(n)=N(n)$.

4.5 Construction

When we have to find a number that satisfies a certain property, the possibilities to approach the problem are, alas, countless. However, theorems of existence provide a bit of solace. For example, the fact that there exist infinitely many primes can be a starting point for the construction of a number with arbitrarily many divisors. Dirichlet's principle (pigeonhole principle) is another powerful theorem of existence, which is, on top of all, non-algorithmic (in other words, it states a fact of existence without giving an algorithm for finding the object we are concerned with.)

4.5.1 Chinese theorem of remainders

The following is a powerful statement of existence and should be borne in mind whenever some construction is needed.

Theorem 4.5.1. *For any set of mutually co-prime positive integers $a_1, a_2, \ldots, a_n$ and any integers $m_1, m_2, \ldots, m_n$ there exists an integer m satisfying*

$$m \equiv m_i \pmod{a_i}$$

for $i = 1, 2, \ldots, n$.

4.5.2 Exercises

Problem 4.5.1. (IMO, 2000) Does there exist a positive integer n such that n has exactly 2000 prime divisors and n divides $2^n + 1$?

Problem 4.5.2. Let $a > 1$ be a positive integer. Prove that for every non-zero positive integer N there exists a positive integer n such that N divides $\left\lfloor \frac{a^n}{n} \right\rfloor$.

Problem 4.5.3. (Singapore Training, 2009) Determine all pairs (p, n) such that p is prime, $n \leq 2p$, and $(p-1)^n + 1$ is divisible by n^{p-1}.

Problem 4.5.4. (IMO SL, 2001)(*) Is it possible to find 100 positive integers not exceeding 25000, such that all pairwise sums of them are different?

Problem 4.5.5. (IMO, 2008) Prove that there are infinitely many positive integers n such that $n^2 + 1$ has a prime divisor greater than $2n + \sqrt{2n}$.

Problem 4.5.6. (IMO, 2003) Let p be a prime number. Prove that there exists a prime number q such that for every integer n, the number $n^p - p$ is not divisible by q.

Problem 4.5.7. (Balkan MO, 2000) Show that for any n we can find a set X of distinct integers greater than 1, such that the average of the elements of any subset of X is a square, cube or higher power.

Problem 4.5.8. (IMO, 1998) For any positive integer n, let $\tau(n)$ denote the number of its positive divisors (including 1 and itself). Determine all positive integers m for which there exists a positive integer n such that

$$\frac{\tau(n^2)}{\tau(n)} = m.$$

Problem 4.5.9. (IMO SL, 1998) Prove that for each positive integer n, there exists a positive integer with the following properties:

(i) it has exactly n digits, none of which is 0;

(ii) it is divisible by the sum of its digits.

Problem 4.5.10. (IMO SL, 1999) Let n, k be positive integers such that n is not divisible by 3 and $k \geq n$. Prove that there exists a positive integer m which is divisible by n and the sum of its digits in decimal representation is k.

Problem 4.5.11. (IMO SL, 1999)(*) Prove that for every real number M there exists an infinite arithmetic progression such that:

(i) each term is a positive integer and the common difference is *not* a multiple of 10;

(ii) the sum of the digits of each term (in decimal representation) exceeds M.

Problem 4.5.12. (Moskow Olympiad?) A sequence of length n that contains n consecutive integers in some order is called an *n-stretch*. For example, $5, 3, 2, 4$ is a four-stretch. Addition of two n-stretches is defined as the sequence whose ith element is the sum of the ith elements of the stretches. For which n do there exist two n-stretches whose sum is also an n-stretch?

Problem 4.5.13. Determine all n for which the divisors of 2001^n (including 1 and itself) can be split into triplets with equal product of the three numbers in each triplet.

Problem 4.5.14. (Kvant, 1988) Let n, m, k be positive integers such that $m \geq n$ and

$$1 + 2 + 3 + \cdots + n = mk.$$

Prove that the numbers $1, 2, 3, \ldots, n$ can be split into k groups so that the sum of the numbers in each group be m.

4.6 Backward induction

4.6.1 Continued fractions

Definition 4.6.1. A fraction of the form

$$a = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}},$$

where a_i are integers and (except possibly a_0) positive, is called *continued fraction* and can be shorthanded as $[a_0, a_1, a_2, \ldots]$. We call the continued fraction *finite* or *infinite* depending on whether the sequence $a_0, a_1, \ldots$ is finite or infinite.

Proposition 4.6.1. Every real number can be uniquely represented as a continued fraction. The continued fraction is finite iff the number is rational.

4.6.2 Pell's equations

Definition 4.6.2. For every D which is not a perfect square, *Pell's equation* is defined as follows:

$$x^2 - Dy^2 = 1, \tag{4.1}$$

where x and y are positive integers.

Before we proceed any further, notice that

- If D is a positive integer but not a perfect square, then $\sqrt{D}$ is irrational;
- If a, b, c, d are integers and α is irrational, then $a + \alpha b = c + \alpha d$ implies that $a = c$ and $b = d$.

Pell's equations provide the backbone of a more generic class of equations $x^2 - Dy^2 = M$. While these equations are quite boring when D is a perfect square (as the left side factors and it all comes down to representing M as a product, which can only happen in finitely many ways,) the case when D is not a perfect square, i.e. when $\sqrt{D}$ is irrational, is quite interesting.

Proposition 4.6.2. There exist infinitely many solutions to Pell's equation 4.1. Moreover, if (x_1, y_1) is the solution for which $x + \sqrt{D}y$ is minimal, then all solutions (x_n, y_n) are given by

$$x_n + \sqrt{D}y_n = \left(x_1 + \sqrt{D}y_1\right)^n.$$

The general case is not much different, except that it might not have a solution.

Proposition 4.6.3. If D is a positive integer but not a perfect square, and the equation $x^2 - Dy^2 = M$ has at least one solution in positive integers, then all solutions are given by

$$x_n + \sqrt{D}y_n = \left(x_0 + \sqrt{D}y_0\right)\left(x_1 + \sqrt{D}y_1\right)^n,$$

where (x_0, y_0) and (x_1, y_1) are the minimal solutions in terms of $x + \sqrt{D}y$ of the equations $x^2 - Dy^2 = M$ and $x^2 - Dy^2 = 1$ respectively.

4.6.3 Exercises

Problem 4.6.1. (Singapore Training, 2009) Let $\alpha > n > 1$ be integers. Prove that the equaiton

$$x_1^2 + x_2^2 + \cdots + x_n^2 = \alpha x_1 x_2 \cdots x_n$$

has no integer solutions other than $(0, 0, \ldots, 0)$.

Problem 4.6.2. (IMO, 1988) Let a and b be two positive integers such that $ab + 1$ divides $a^2 + b^2$. Show that $\dfrac{a^2 + b^2}{ab + 1}$ is a perfect square.

Problem 4.6.3. (Bulgarian BO selection) If m, n, and k are positive integers such that $\dfrac{m^2+n^2}{mn-1} = k$, prove that $k = 5$.

Problem 4.6.4. (IMO, 2007) Let a and b be positive integers. Show that if $4ab-1$ divides $(4a^2-1)^2$, then $a = b$.

4.7 Unsorted

Problem 4.7.1. (Kvant, 1990) A chess tournament, in which n players played against each other, is called *logical* if no player A beat another player who scored at least as much as him/her in the final ranking (for a win there is 1 point, for a draw there is a half point, and for a loss there are no points.) Prove that no matter the final results, it would have been possible to achieve the same final results (distribution of points among the contestants) in a logical tournament.

Problem 4.7.2. If a and b are positive integers such that

$$\frac{a^2}{(2a-b)b^2+1}$$

is an integer, prove that either $b = 1$ or $b = 2a$.

Problem 4.7.3. (Kvant 1976)

(i) Using only the digits 1 and 2, prove that one can write 2^{n+1} numbers, each with 2^n digits, so that every two differ in at least 2^{n-1} places.

(ii) Prove that onne cannot find more than 2^{n+1} such numbers with 2^n digits.

Problem 4.7.4. (Singapore MO, 2006) Let n be a positive integer and let $S_1, S_2, \ldots, S_n$ be a collection of $2n$-element subsets of $\{1, 2, 3, \ldots, 4n-1, 4n\}$ so that $S_i \cap S_j$ contains at most n elements for all $i \neq j$. Show that

$$k \leq 6^{\frac{n+1}{2}}.$$

Problem 4.7.5. (Hypothesis) Given are n vectors, each of length less than 1, which sum up to 0. Prove that they can be ordered as $a_1, a_2, \ldots, a_n$ in such a way that for each $k = 1, 2, \ldots, n$ be true:

$$\left|\sum_{i=1}^{k} a_i\right| < 1.$$

Problem 4.7.6. (Singapore Training, 2009) Prove that for any real x and positive integer n,

$$\lfloor nx \rfloor \geq \frac{\lfloor x \rfloor}{1} + \frac{\lfloor 2x \rfloor}{2} + \frac{\lfloor 3x \rfloor}{3} + \cdots + \frac{\lfloor nx \rfloor}{n}.$$

Chapter 5

Hints to the exercises

5.1 Algebra

1.5.7 Show that for any k, there is an $n > k$ such that $f(n) \leq k + 1$.

5.2 Geometry

2.4.7 Project KN and OM onto a line.

5.3 Combinatorics

3.2.5 Consider the polynomial $\binom{xn-1}{n-1}$.

3.10.9 Consider the functions $\sinh(x)$ and $\sinh(2x)$.

3.3.22 Prove that $\sum \left(F_1^{p-1} + F_2^{p-1} + \cdots + F_{p-1}^{p-1}\right)^{p-1}$ over all $x_1, x_2, \ldots, x_{(p-1)^2+1}$ in $\mathbb{Z}_p$ is congruent to 0, where $F_i = \sum_{j=1}^{(p-1)^2+1} a_{ji} x_j^{p-1}$ for $i = 1, 2, \ldots, p-1$.

3.5.5 Consider $\mathbb{Z}_2[\omega]$ where $\omega^3 = 1$.

3.8.12 Put all in the same room and consider sending members of the largest clique over to the other room.

5.4 Number Theory

4.1.4 Consider the numbers $\left\lfloor \frac{p}{k} \right\rfloor$ for $k = 2, 3, \ldots$.
4.3.5 Consider a polynomial in $\mathbb{Z}_p$.