

# Certain applications of the roots of unity

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## Abstract

The objective of this article is to illustrate the use of the roots of unity for solving problems from various mathematical areas, like combinatorics, algebra, number theory. The selected examples are either original or from national and international mathematical competitions.

It is well known that for every natural number  $n$  the equation

$$x^n - 1 = 0$$

has  $n$  distinct complex roots  $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1}$ , where

$$\varepsilon_j = \cos\left(\frac{2\pi j}{n}\right) + i \sin\left(\frac{2\pi j}{n}\right), \quad j = 0, 1, \dots, n-1.$$

The complex numbers  $\varepsilon_j$  have modulus 1 and if we set

$$\varepsilon = \cos\left(\frac{2\pi}{n}\right) + i \sin\left(\frac{2\pi}{n}\right),$$

then from De Moivre's formula it follows that  $\varepsilon_j = \varepsilon^j$ . Furthermore  $\varepsilon^k = \varepsilon^l$  for integers  $k$  and  $l$  if and only if  $k \equiv l \pmod{n}$ .

The following property of roots of unity is frequently used and therefore we isolate it as:

**Lemma.** For every positive integer  $n$ ,

$$\sum_{j=0}^{n-1} \varepsilon_j^k = \begin{cases} n, & \text{if } k \equiv 0 \pmod{n} \\ 0, & \text{if } k \not\equiv 0 \pmod{n} \end{cases} \quad (*)$$

**Proof.** Notice that  $\varepsilon^k = 1$  if and only if  $k \equiv 0 \pmod{n}$ . Therefore

$$\sum_{j=0}^{n-1} \varepsilon_j^k = \sum_{j=0}^{n-1} (\varepsilon^k)^j = \begin{cases} \frac{\varepsilon^{kn} - 1}{\varepsilon^k - 1} = \frac{1-1}{\varepsilon^k - 1} = 0, & \text{if } k \not\equiv 0 \pmod{n}, \\ 1 + 1 + \dots + 1 = n, & \text{if } k \equiv 0 \pmod{n}. \end{cases}$$

Let us note that roots of unity can be represented geometrically as the vertices of a regular  $n$ -gon inscribed in the unit circle.

We begin with two problems where exactly this geometric interpretation is used.

**Problem 1. (USSR Olympiad 1970)** The vertices of a regular  $n$ -gon are colored in some colors (each vertex in one color) such that the points of one and the same color are vertices of regular polygons. Prove that among these polygons there are two identical ones.

**Solution.** Without loss of generality we may consider the vertices of the  $n$ -gon as  $n$ -th roots of unity. Suppose that no two monochromatic polygons are identical and let them have respectively  $d_1 < d_2 < \dots < d_m$  vertices.

Let us sum the  $d_1$ -th powers of the “vertices” of these polygons. From  $n > d_1$  it follows that  $n$  does not divide  $d_1$ , and from (\*) we conclude that this sum is 0.

But if we compute this sum separately for each of the monochromatic polygons, then from  $d_j > d_1$  for  $j = 2, 3, \dots, m$  and  $d_1 \mid d_j$  it follows by (\*) that it equals

$$\varepsilon^k d_1 + 0 + \dots + 0 = d_1 \varepsilon^k \neq 0$$

for some integer  $k$ . The obtained contradiction proves the statement.  $\square$

**Problem 2. (USAMO)** Points  $A_1, A_2, \dots, A_n$  are chosen on the unit circle. Prove that if

$$|MA_1| \cdot |MA_2| \cdots |MA_n| \leq 2$$

for every point  $M$  on the circle, then  $A_1, A_2, \dots, A_n$  are vertices of a regular  $n$ -gon.

**Solution.** Consider the polynomial

$$f(z) = (z - a_1)(z - a_2) \cdots (z - a_n),$$

where  $a_j$  is the affix of point  $A_j$ . Notice that if  $M$  has affix  $z$  on the unit circle, then

$$|MA_1| \cdot |MA_2| \cdots |MA_n| = |f(z)|.$$

Let

$$a = (-1)^n (a_1 a_2 \cdots a_n)^{-1}, \quad \tau = a^{1/n}, \quad |a| = |\tau| = 1.$$

Set

$$g(z) = af\left(\frac{z}{\tau}\right) = z^n + A_{n-1}z^{n-1} + \cdots + A_1z + 1.$$

If we assume  $|f(z)| \leq 2$  for every  $z : |z| = 1$ , then also  $|g(z)| \leq 2$  for every  $z : |z| = 1$ . Then we obtain

$$2n = \left| \sum_{j=0}^{n-1} g(\varepsilon_j) \right| \leq \sum_{j=0}^{n-1} |g(\varepsilon_j)| \leq 2n.$$

This is possible only when  $g(\varepsilon_j) = 2$  for  $j = 0, 1, \dots, n-1$ .

This means that  $g(z) - 2 = z^n - 1$  and therefore the equation  $f(z) = 0$  has roots

$$\frac{1}{\tau} \left( \cos \left( \frac{(2j-1)\pi}{n} \right) + i \sin \left( \frac{(2j-1)\pi}{n} \right) \right),$$

for  $j = 1, 2, \dots, n$ , which are vertices of a regular  $n$ -gon.  $\square$

**Problem 3. (MOM'95 Canada)** For a given prime  $p > 2$  determine the number of  $p$ -element subsets of  $\{1, 2, \dots, 2p\}$  such that the sum of their elements is divisible by  $p$ .

**Solution.** Consider the polynomial

$$f(x, y) = (1 + yx)(1 + yx^2)(1 + yx^3) \cdots (1 + yx^{2p}).$$

It can be written in the form

$$f(x, y) = q_0(x) + q_1(x)y + \cdots + q_{2p}(x)y^{2p},$$

with  $q_k(x) = \sum x^{\alpha_1 + \cdots + \alpha_k}$ , where the sums are taken over all  $k$ -element subsets of  $\{1, 2, \dots, 2p\}$ . Now let

$$q_p(x) = A_0 + A_1x + A_2x^2 + \cdots,$$

then the sought number is

$$A = A_0 + A_p + A_{2p} + \cdots.$$

From (\*) we have

$$\sum_{j=0}^{p-1} f(x, \varepsilon_j) = p(q_0(x) + q_p(x) + q_{2p}(x)) = p(1 + x^{p(2p+1)}) + pq_p(x).$$

But

$$\sum_{j=0}^{p-1} q_p(\varepsilon_j) = p(A_0 + A_p + \cdots) = pA,$$

so

$$\sum_{j,k=0}^{p-1} f(\varepsilon_j, \varepsilon_k) = p(p+p) + p^2 A = p^2(A+2).$$

We compute the same sum:

$$\begin{aligned} \sum_{j,k=0}^{p-1} f(\varepsilon_j, \varepsilon_k) &= \sum_{k=0}^{p-1} f(1, \varepsilon_k) + \sum_{j=1}^{p-1} \sum_{k=0}^{p-1} f(\varepsilon_j, \varepsilon_k) \\ &= \sum_{k=0}^{p-1} (1 + \varepsilon_k)^{2p} + p(p-1) \prod_{j=0}^{p-1} (1 + \varepsilon_j)^2. \end{aligned}$$

We know that

$$g(x) = x^p - 1 = \prod_{j=0}^{p-1} (x - \varepsilon_j),$$

thus

$$4 = g^2(-1) = \prod_{j=0}^{p-1} (1 + \varepsilon_j)^2,$$

and from (\*) we have

$$\sum_{j=0}^{p-1} (1 + \varepsilon_j)^{2p} = p \binom{2p}{0} + p \binom{2p}{p} + p \binom{2p}{2p} = p \left( 2 + \binom{2p}{p} \right).$$

Thus

$$p^2(A+2) = 4p^2 - 2p + p \binom{2p}{p}$$

and

$$A = \frac{\binom{2p}{p} - 2}{p} + 2.$$

**Problem 4. (Author)** In a table  $(a_{kl})$  of size  $6 \times 21$  are written 18 zeros, 18 ones, ..., 18 sixes. For arbitrary  $k, l \in [1, 6]$  and  $p, q \in [1, 21]$  it holds

$$a_{kp} - a_{lp} \equiv a_{kq} - a_{lq} \pmod{7}.$$

Show that in each row there are exactly 3 zeros, 3 ones, ..., 3 sixes.

**Solution.** Construct a new table  $(b_{kp})$  where

$$b_{kp} = \varepsilon^{a_{kp}}, \quad \varepsilon = \cos\left(\frac{2\pi}{7}\right) + i \sin\left(\frac{2\pi}{7}\right).$$

Let the sum of numbers in the  $k$ -th row be  $s_k$ . From the condition we have  $s_k = s_1 \varepsilon^{m_k}$  for some integers  $m_k$ , which we may consider to be integers between 0 and 6. Let

$$q(x) = 1 + x + \cdots + x^6.$$

Then

$$s_1(\varepsilon^{m_1} + \varepsilon^{m_2} + \cdots + \varepsilon^{m_6}) = \sum_{k=1}^6 s_k = 18q(\varepsilon) = 0.$$

Let

$$p(x) = x^{m_1} + x^{m_2} + \cdots + x^{m_6}$$

and

$$s(x) = \sum_{p=1}^{21} x^{a_{1p}}.$$

Then  $s(\varepsilon) = s_1$ . Hence  $s(\varepsilon)p(\varepsilon) = 0$ , hence  $\varepsilon$  is a root of either  $s(x)$  or  $p(x)$ . We shall use the well-known fact that  $q(x)$  is irreducible over  $\mathbb{Q}$ .

If  $p(\varepsilon) = q(\varepsilon) = 0$ , it follows that  $q(x) \mid p(x)$ . Since  $\deg(p) \leq 6 = \deg(q)$ , we get  $p(x) = kq(x)$  for some integer  $k$ . But then  $7k = kq(1) = p(1) = 6$ , which is impossible.

Therefore,  $s(\varepsilon) = q(\varepsilon) = 0$ . Using the irreducibility of  $q$  we get  $s(x) = kq(x)$ . From  $21 = s(1) = kq(1) = 7k$  we get  $k = 3$ , hence

$$s(x) = 3 + 3x + \cdots + 3x^6,$$

which shows that the first row of the table contains 3 zeros, 3 ones,  $\dots$ , 3 sixes. The same holds true for each of the other rows, with which the problem is solved.  $\square$

**Problem 5. (Leningrad Olympiad 1991)** A sequence  $a_0, a_1, \dots, a_{n-1}$  is called  $p$ -balanced if all sums of the form

$$a_k + a_{k+p} + a_{k+2p} + \cdots$$

for  $k = 0, 1, \dots, p - 1$  are equal.

Prove that if a sequence of 50 terms is  $p$ -balanced for  $p = 3, 5, 7, 11, 13, 17$ , then all terms are equal to 0.

**Solution.** Let the sequence be  $a_0, a_1, \dots, a_{49}$ . Form the polynomial

$$q(x) = \sum_{j=0}^{49} a_j x^j.$$

Take the prime  $p = 13$  for starters and let  $\varepsilon = \cos\left(\frac{2\pi}{p}\right) + i \sin\left(\frac{2\pi}{p}\right)$ . Then for  $k = 1, 2, \dots, p - 1$  we have

$$\begin{aligned} q(\varepsilon^k) &= (a_0 + a_p + \dots) + (a_1 + a_{p+1} + \dots)\varepsilon^k + (a_2 + a_{p+2} + \dots)\varepsilon^{2k} + \dots \\ &= (a_0 + a_p + \dots) \left(1 + \varepsilon^k + \varepsilon^{2k} + \dots + \varepsilon^{(p-1)k}\right) \\ &= 0 \end{aligned}$$

owing to (\*), so  $\varepsilon^k, k = 1, 2, \dots, p - 1$  gives us  $p - 1 = 12$  distinct roots of  $q(x)$ . Similarly, using the other primes 3, 5, 7, 11, 17 we obtain that  $q(x)$  has  $(3 - 1) + (5 - 1) + (7 - 1) + (11 - 1) + (13 - 1) + (17 - 1) = 50$  distinct roots, since different primes give rise to disjoint sets of  $\varepsilon_k$ . From  $\deg q = 49$ , it follows that  $q(x) \equiv 0$ . Thus

$$a_0 = a_1 = \dots = a_{49} = 0$$

and the proof is complete.  $\square$

**Problem 6. (IMO 1999 shortlist)** Let  $p > 5$  be a prime number. Denote by  $E(0, 1, 2)$  the number of all sums of the form

$$x_1 + 2x_2 + \dots + (p - 1)x_{p-1},$$

which are divisible by  $p$ , where  $x_i \in \{0, 1, 2\}$ . Analogously define the number  $E(0, 1, 3)$ . Prove that

$$E(0, 1, 2) < E(0, 1, 3).$$

**Solution:** Let us consider the polynomial

$$f(x) = (1 + x + x^2)(1 + x^2 + x^4) \dots \left(1 + x^{p-1} + x^{2(p-1)}\right).$$

This can be written in the form

$$f(x) = \frac{(x^3 - 1)(x^6 - 1) \dots (x^{3(p-1)} - 1)}{(x - 1)(x^2 - 1) \dots (x^{p-1} - 1)}.$$

As usual, we set

$$\varepsilon_j = \cos\left(\frac{2\pi j}{p}\right) + i \sin\left(\frac{2\pi j}{p}\right), \quad j = 0, 1, \dots, p-1$$

and denote  $\varepsilon = \varepsilon_1$ . From (\*) it follows that

$$pE(0, 1, 2) = \sum_{j=0}^{p-1} f(\varepsilon_j).$$

Observe that the following two sets are the same:

$$\{\varepsilon_0^3, \varepsilon_1^3, \dots, \varepsilon_{p-1}^3\} = \{\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{p-1}\}.$$

Then for  $j = 1, 2, \dots, p-1$  we have:

$$f(\varepsilon_j) = \frac{(\varepsilon_j^3 - 1)(\varepsilon_j^6 - 1) \cdots (\varepsilon_j^{3(p-1)} - 1)}{(\varepsilon_j - 1)(\varepsilon_j^2 - 1) \cdots (\varepsilon_j^{p-1} - 1)} = 1.$$

Therefore

$$pE(0, 1, 2) = 3^{p-1} + p - 1.$$

Analogously we consider the polynomial

$$g(x) = (1 + x + x^3)(1 + x^2 + x^6) \cdots (1 + x^{p-1} + x^{3(p-1)}).$$

Again

$$pE(0, 1, 3) = \sum_{j=0}^{p-1} g(\varepsilon_j) = g(1) + \sum_{j=1}^{p-1} g(\varepsilon_j) = 3^{p-1} + (p-1)g(\varepsilon),$$

since  $g(\varepsilon) = g(\varepsilon_1) = g(\varepsilon_2) = \cdots = g(\varepsilon_{p-1})$ . It remains to prove that

$$g(\varepsilon) > f(\varepsilon) = 1.$$

We have

$$\begin{aligned}
g(\varepsilon) &= \prod_{j=1}^{p-1} (1 + \varepsilon^j + \varepsilon^{3j}) = \prod_{j=1}^{\frac{p-1}{2}} (1 + \varepsilon^j + \varepsilon^{3j})(\bar{1} + \bar{\varepsilon}^j + \bar{\varepsilon}^{3j}) = \\
&= \prod_{j=1}^{\frac{p-1}{2}} \left( 3 + (\varepsilon^j + \bar{\varepsilon}^j) + (\varepsilon^{2j} + \bar{\varepsilon}^{2j}) + (\varepsilon^{3j} + \bar{\varepsilon}^{3j}) \right) \\
&= \prod_{j=1}^{\frac{p-1}{2}} \left( 3 + 2 \cos\left(\frac{2\pi j}{p}\right) + 2 \cos\left(\frac{4\pi j}{p}\right) + 2 \cos\left(\frac{6\pi j}{p}\right) \right) \\
&= \prod_{j=1}^{\frac{p-1}{2}} q(\cos(\alpha_j)),
\end{aligned}$$

where

$$\alpha_j = \frac{2\pi j}{p}$$

and

$$q(x) = 3 + 2x + 2(2x^2 - 1) + 2(4x^3 - 3x) = 8x^3 + 4x^2 - 4x + 1.$$

More detailed investigation of  $q(x) = q(\cos(\alpha))$  and its extrema shows that:

1.  $q(x) > 1$ , for  $x \in (-1, 0) \leftrightarrow \alpha \in \left(\frac{\pi}{2}, \pi\right)$ ;
2.  $q(x) \geq \frac{47 - 14\sqrt{7}}{27} > \frac{1}{3}$ , for  $x \in \left[0, \frac{1}{2}\right] \leftrightarrow \alpha \in \left[\frac{\pi}{3}, \frac{\pi}{2}\right]$ ;
3.  $q(x) \geq 1$ , for  $x \in \left[\frac{1}{2}, \frac{\sqrt{2}}{2}\right] \leftrightarrow \alpha \in \left[\frac{\pi}{4}, \frac{\pi}{3}\right]$ ;
4.  $q(x) \geq 3$ , for  $x \in \left[\frac{\sqrt{2}}{2}, 1\right] \leftrightarrow \alpha \in \left[0, \frac{\pi}{4}\right]$ .

Consider those four intervals for  $\alpha_j$ . When  $\alpha_j$  lies in the first or third interval,  $q(\cos(\alpha_j)) \geq 1$ ; when it lies in the second interval,  $q(\cos(\alpha_j)) > \frac{1}{3}$ ; and when it lies in the fourth interval,  $q(\cos(\alpha_j)) \geq 3$ . There is at least one  $\alpha_j$  in the first interval where  $q(\cos(\alpha_j)) > 1$ , e.g. for  $j = \frac{p-1}{2}$  as long as

$p > 2$ . So to prove the strict inequality  $g(\varepsilon) > 1$ , it remains to prove that the number of the  $\alpha_j$  in the second interval does not exceed the number of the  $\alpha_j$  in the fourth interval. Those counts are respectively

$$\left\lfloor \frac{p}{4} \right\rfloor - \left\lfloor \frac{p}{6} \right\rfloor \quad \text{and} \quad \left\lfloor \frac{p}{8} \right\rfloor.$$

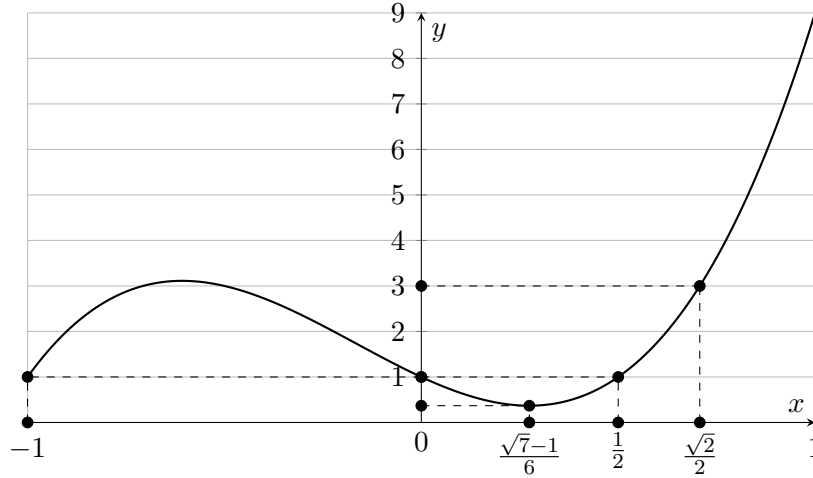
We need to show that for every prime  $p > 5$

$$\left\lfloor \frac{p}{8} \right\rfloor \geq \left\lfloor \frac{p}{4} \right\rfloor - \left\lfloor \frac{p}{6} \right\rfloor.$$

For  $p = 7, 11, 13, 17, 19, 23, 29, 31$  this is checked directly. For  $p \geq 35$  we have

$$\left\lfloor \frac{p}{8} \right\rfloor \geq \frac{p-7}{8} \geq \frac{p-1}{4} - \frac{p-5}{6} \geq \left\lfloor \frac{p}{4} \right\rfloor - \left\lfloor \frac{p}{6} \right\rfloor$$

and the proof is complete.  $\square$



The next problem is original. It was not in the first version of the article from 2002, as the number 2026 gives it away, but I added it here as it fits nicely with the topic.

**Problem 7.** Let  $n = 2026$ . Given a regular  $n$ -gon  $A_1A_2 \cdots A_n$  with side length 1, prove that one can choose a subset of the vectors

$$\overrightarrow{A_1A_2}, \overrightarrow{A_2A_3}, \dots, \overrightarrow{A_nA_1}$$

whose sum is a nonzero vector of length less than

$$\frac{2n}{5 \left( 2^{\frac{n-2}{4}} - 1 \right)}.$$

**Solution.** Write  $n = 2p$ , where  $p = 1013$  is an odd prime. Identify the plane of the polygon with the complex plane in such a way that the side vectors

$$\overrightarrow{A_1A_2}, \overrightarrow{A_2A_3}, \dots, \overrightarrow{A_nA_1}$$

correspond to

$$1, \varepsilon, \varepsilon^2, \dots, \varepsilon^{n-1},$$

where

$$\varepsilon = e^{2\alpha i}, \quad \alpha = \frac{\pi}{n}.$$

It suffices to find a nonempty subset of  $\{1, \varepsilon, \dots, \varepsilon^{n-1}\}$  whose sum is nonzero and suitably small. Let  $m = \frac{p-1}{2}$ , and for  $j = 0, 1, \dots, m-1$ , define

$$q_j = \{\varepsilon^j, \varepsilon^{p-j-1}\}, \quad t_j = \{\varepsilon^{p+j}, \varepsilon^{2p-j-1}\}.$$

These  $2m = p-1$  groups are pairwise disjoint. Also, each element of  $q_j$  is opposite to the corresponding element of  $t_j$ , since

$$\varepsilon^j = -\varepsilon^{p+j}, \quad \varepsilon^{p-j-1} = -\varepsilon^{2p-j-1}.$$

For each integer  $j$ , let  $B_j$  be the point on the unit circle corresponding to the complex number  $\varepsilon^j$ , and let  $O$  be the origin. Then each triangle  $\triangle B_jOB_{p-j-1}$  is isosceles, and

$$\angle OB_jB_{p-j-1} = \angle OB_{p-j-1}B_j = (2j+1)\alpha.$$

The internal angle bisector of  $\angle B_jOB_{p-j-1}$  is the same ray for every  $j$ ; denote it by  $\overrightarrow{OU}$ , where  $U$  is the point on the unit circle such that

$$\angle UOB_0 = \frac{\pi}{2} - \alpha.$$

Hence

$$\overrightarrow{OB_j} + \overrightarrow{OB_{p-j-1}} = 2 \sin((2j+1)\alpha) \overrightarrow{OU}.$$

Equivalently, if  $s_j$  denotes the sum of the elements in  $q_j$ , then

$$s_j = 2 \sin((2j+1)\alpha)u,$$

where  $u$  is the unit vector in the direction of  $\overrightarrow{OU}$ .

Let  $M = \{0, 1, \dots, m-1\}$ . We shall show that the  $2^m$  subset sums

$$\sum_{j \in A} s_j, \quad A \subseteq M,$$

are all distinct. Suppose, to the contrary, that

$$\sum_{j \in A'} s_j = \sum_{j \in A''} s_j$$

for two distinct subsets  $A', A'' \subseteq M$ . Removing common elements if necessary, we may assume that  $A'$  and  $A''$  are disjoint. Then  $\varepsilon$  is a root of the polynomial

$$a(x) = \sum_{j \in A'} (x^j + x^{p-1-j}) - \sum_{j \in A''} (x^j + x^{p-1-j}).$$

The minimal polynomial of  $\varepsilon$  over  $\mathbb{Q}$  is

$$b(x) = x^{p-1} - x^{p-2} + \dots - x + 1.$$

Indeed,

$$b(\varepsilon)(1 + \varepsilon) = \varepsilon^p + 1 = 0,$$

and since  $\varepsilon \neq -1$ , we get  $b(\varepsilon) = 0$ . Also,  $b(x)$  is irreducible over  $\mathbb{Q}$  because

$$b(-x) = x^{p-1} + x^{p-2} + \dots + x + 1 = \Phi_p(x),$$

the  $p$ th cyclotomic polynomial. Hence  $b(x)$  is the minimal polynomial of  $\varepsilon$ , so

$$b(x) \mid a(x).$$

Therefore  $\deg a \geq \deg b = p-1$ . On the other hand, since  $0 \leq j \leq m-1$ , we have  $\deg a \leq p-1$ , with equality only if  $0 \in A'$  or  $0 \in A''$ . Without loss of generality, assume  $0 \in A'$ . Then

$$a(x) = (1 + x^{p-1}) + c(x) = b(x) + d(x) + c(x),$$

where

$$c(x) = \sum_{j \in A' \setminus \{0\}} (x^j + x^{p-1-j}) - \sum_{j \in A''} (x^j + x^{p-1-j}),$$

and

$$d(x) = x^{p-2} - x^{p-3} + \dots + x.$$

Both  $c(x)$  and  $d(x)$  have degree at most  $p - 2$ . Since  $b(x) \mid a(x)$ , we have

$$b(x) \mid (c(x) + d(x)).$$

But  $\deg(c+d) \leq p-2$ , so this forces  $c(x) + d(x) \equiv 0$ , which is impossible because  $c(1) = 2(|A'| - 1) - 2|A''|$  is even, while  $d(1) = 1$  is odd. Contradiction.

Thus the  $2^m$  subset sums  $\sum_{j \in A} s_j$  are all distinct. Each has the form  $\lambda u$ , where

$$0 \leq \lambda \leq 2 \sum_{j=0}^{m-1} \sin((2j+1)\alpha) < 2 \sum_{j=0}^{m-1} (2j+1)\alpha = 2m^2\alpha.$$

Hence these  $2^m$  distinct points all lie on a segment of length less than  $2m^2\alpha$ . By the pigeonhole principle, two of them must be at distance less than

$$\Delta = \frac{2m^2\alpha}{2^m - 1}.$$

So there exist distinct subsets  $A', A'' \subseteq M$  such that

$$\left| \sum_{j \in A'} s_j - \sum_{j \in A''} s_j \right| \in (0, \Delta).$$

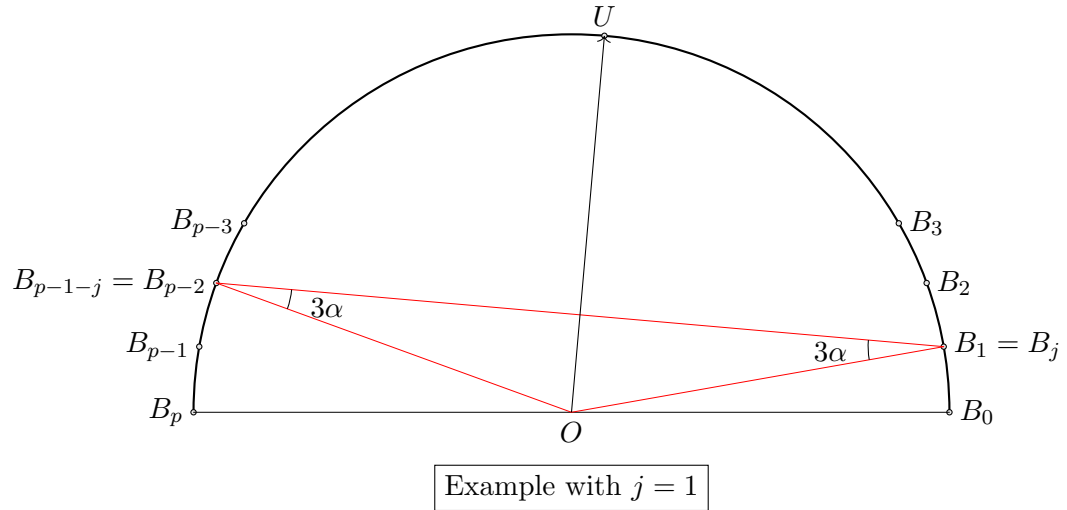
Therefore

$$\left| \sum_{j \in A'} \sum_{x \in q_j} x + \sum_{j \in A''} \sum_{x \in t_j} x \right| = \left| \sum_{j \in A'} s_j - \sum_{j \in A''} s_j \right| \in (0, \Delta).$$

This is the sum of a subset of the original  $n$  vectors. Finally,

$$\Delta = \frac{\pi}{n} \left( \frac{n-2}{4} \right)^2 \frac{2}{2^{(n-2)/4} - 1} = \frac{\pi(n-2)^2}{8n(2^{(n-2)/4} - 1)} < \frac{2n}{5(2^{(n-2)/4} - 1)},$$

since  $\pi < 16/5$  and  $(n-2)^2 < n^2$ . This completes the proof.  $\square$



We shall conclude the article by leaving one last problem for exercise.

**Problem 8.** Given is a regular 2001-gon  $A_0A_1 \dots A_{2000}$  inscribed in the unit circle. Prove that  $|A_0A_1| \cdot |A_0A_2| \cdots |A_0A_{2000}| = 2001$ .