Similar triangles inscribed in one another

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The use of complex numbers in geometry is often considered to be the poor cousin of purely geometric solutions. Yet, despite its lack of aesthetics, a bit of algebra can lead to surprisingly interesting geometric results. In this article, we examine families of triangles inscribed in one another.

Definition 1. We shall call two triangles $\triangle XYZ$ and $\triangle X'Y'Z'$ *similar* and denote this by $\triangle XYZ \sim \triangle X'Y'Z'$ if they have the same orientation and $XY : YZ : ZX = X'Y' : Y'Z' : Z'X'$. (Notice that, under this definition, $\triangle XYZ \not\sim \triangle XZY$ because the orientation is different.)

Definition 2. We shall say that $\triangle X'Y'Z'$ is inscribed in $\triangle ABC$ if

$$X' \in BC, \quad Y' \in CA, \quad Z' \in AB,$$

where AB, BC, CA denote the *lines* and not just the segments.

1 General form of the transformation

Consider two arbitrary triangles $\triangle ABC$ and $\triangle XYZ$ inscribed in the unit circle in the complex plane. Let $\Phi_{\triangle ABC}(\triangle XYZ) = \Phi$ be the set of all triangles inscribed in $\triangle ABC$ and similar to $\triangle XYZ$.

For every triangle $\triangle X'Y'Z' \in \Phi$, there is a similarity transformation that sends $\triangle XYZ$ into $\triangle X'Y'Z'$, which can be described as an affine map $\rho(w) = uw + v$ in the complex plane. We have

$$\rho(x) = ux + v = x'$$
$$\rho(y) = uy + v = y'$$
$$\rho(z) = uz + v = z'$$

From $X' \in BC$, we have

$$\frac{x'-b}{b-c} = \frac{\overline{x'-b}}{\overline{b-c}} \Rightarrow \frac{ux+v-b}{b-c} = bc\frac{\overline{ux}+\overline{v}-\overline{b}}{c-b} \Rightarrow ux+\overline{u}\frac{bc}{x}+v+\overline{v}bc = b+c \tag{1}$$

Similarly, from $Y' \in CA$ it follows that $uy+\overline{u}\frac{ca}{y}+v+\overline{v}ca = c+a$. Subtracting from (1), we get

$$u(x-y)+\overline{u}\left(\frac{bc}{x}-\frac{ca}{y}\right)+\overline{v}c(b-a) = b-a \tag{2}$$

In the same way, we derive

$$u(y-z)+\overline{u}\left(\frac{ca}{y}-\frac{ab}{z}\right)+\overline{v}a(c-b) = c-b \tag{3}$$

Dividing (2) and (3) by $c(b-a)$ and $a(c-b)$ respectively and subtracting from one another, we get

$$u\left(\frac{ax(c-b)+by(a-c)+cz(b-a)}{ac(b-a)(c-b)}\right)+\overline{u}\left(b\frac{xy(a-b)+yz(b-c)+zx(c-a)}{xyz(a-b)(b-c)}\right) = \frac{a-c}{ac},$$

which simplifies to

$$ut+\overline{ut} = -1, \tag{4}$$

where

$$t = \frac{ax(c-b)+by(a-c)+cz(b-a)}{(a-b)(b-c)(c-a)}. \tag{5}$$

A necessary condition for $\Phi \neq \emptyset$ is $t \neq 0$.
Now, if we conjugate both sides of (2), we get

$$\overline{u}\frac{abc(y-x)}{xy(a-b)}+u\frac{ax-by}{a-b}+v = c.$$

Plugging in $\overline{u} = \dfrac{-1-ut}{\overline{t}}$ from (4) into the previous equation and after simplifying, we get

$$up+v = q, \tag{6}$$

where

$$p = \frac{ax-by}{a-b}-\frac{t}{\overline{t}}\frac{(y-x)abc}{(a-b)xy}, \tag{7}$$

$$q = c+\frac{1}{\overline{t}}\frac{(y-x)abc}{(a-b)xy}. \tag{8}$$

After simplifying (7) and (8), we get

$$p = \frac{ax^2(z-y) + by^2(x-z) + cz^2(y-x)}{yz(c-b) + zx(a-c) + xy(b-a)}, \tag{9}$$

$$q = \frac{xyc(b-a) + yza(c-b) + zxb(a-c)}{yz(c-b) + zx(a-c) + xy(b-a)}. \tag{10}$$

Therefore, from (4) and (6), we get

$$\rho(w) = \frac{\mu}{t}(w-p) + q, \tag{11}$$

where μ is a complex number with real part $-\frac{1}{2}$. It is easy to see that the image of any given point under ρ is a line passing through $q - \frac{w-p}{2t}$ and perpendicular to the vector $\frac{w-p}{t}$.

2 Examining the set Φ

Proposition 1. The set Φ

(i) is the empty set when $\triangle XYZ$ is similar to the mirror image of $\triangle ABC$.

(ii) contains infinitely many triangles, namely the images of $\triangle XYZ$ under the transformation (11) for all μ with real part $-\frac{1}{2}$ when $\triangle XYZ$ is not similar to the mirror image of $\triangle ABC$.

Proof: We shall show first that $t = 0$ is equivalent to $(x, y, z) = \lambda(\bar{a}, \bar{b}, \bar{c})$ for some $\lambda \neq 0$, which in turn is equivalent to $\triangle XYZ$ being similar to the mirror image of $\triangle ABC$.

The forward implication is easy to verify by plugging in $x = \lambda\bar{a}, y = \lambda\bar{b}$, and $z = \lambda\bar{c}$ into (5).

Now, consider $t = 0$. From (5), we get

$$\begin{aligned} x(\bar{c}-\bar{b}) + y(\bar{a}-\bar{c}) + z(\bar{b}-\bar{a}) &= t\frac{(a-b)(b-c)(c-a)}{abc} \\ &= 0, \end{aligned}$$

which can be re-written as

$$\begin{vmatrix} x & y & z \\ 1 & 1 & 1 \\ \bar{a} & \bar{b} & \bar{c} \end{vmatrix} = 0. \tag{12}$$

As the vectors $(1,1,1)$ and $(\bar{a},\bar{b},\bar{c})$ are linearly independent, the last equation is equivalent to

$$(x,y,z) = \lambda_1(1,1,1) + \lambda_2(\bar{a},\bar{b},\bar{c})$$

for some complex λ_1 and λ_2. From here and $|a| = |b| = |c| = 1$, we get

$$|x - \lambda_1| = |y - \lambda_1| = |z - \lambda_1| = |\lambda_2|,$$

whence the *three different* points X, Y, Z must lie on a circle centered at λ_1 with radius $|\lambda_2|$ *and at the same time* lie on the unit circle. The two circles can have three points in common only when they coincide, i.e. when $\lambda_1 = 0$ and $|\lambda_2| = 1$, from where $(x,y,z) = \lambda(\bar{a},\bar{b},\bar{c})$.

Now, we want to show that if $t \neq 0$, every transformation (11) with $\mathrm{Re}(\mu) = -\dfrac{1}{2}$ sends $\triangle XYZ$ into a similar $\triangle X'Y'Z'$ inscribed in $\triangle ABC$. We shall only show that $X' \in BC$, as the other two conditions follow analogously. The condition $X' \in BC$ is equivalent to

$$\frac{x' - b}{b - c} = \frac{\overline{x' - b}}{\overline{b - c}},$$

where $x' = \rho(x) = \dfrac{\mu}{t}(x - p) + q$ and $\mu + \bar{\mu} = -1$. Simplifying the last equation and replacing $\bar{\mu}$ by $-1 - \mu$, it remains to prove

$$\mu\left(\frac{x - p}{t} - \frac{\bar{x} - \bar{p}}{\bar{t}}bc\right) + \left(q + \bar{q}bc - b - c - \frac{\bar{x} - \bar{p}}{\bar{t}}bc\right) = 0. \tag{13}$$

We shall show first that

$$q + \bar{q}bc - b - c = \frac{x - p}{t}. \tag{14}$$

From (8) and (7), we have

$$\begin{aligned} q + \bar{q}bc - b - c &= (q - c) + bc(\overline{q - c}) \\ &= \frac{1}{\bar{t}}\frac{(y - x)abc}{(a - b)xy} + bc\frac{1}{t}\frac{(y - x)}{(a - b)c} \\ &= \frac{1}{t}\left(x - \frac{ax - by}{a - b} + \frac{t}{\bar{t}}\frac{(y - x)abc}{(a - b)xy}\right) \\ &= \frac{x - p}{t}, \end{aligned}$$

which proves (14). Therefore,

$$\frac{\bar{x} - \bar{p}}{\bar{t}}bc = (\bar{q} + q\overline{bc} - \bar{b} - \bar{c})bc = \bar{q}bc + q - b - c = \frac{x - p}{t},$$

which shows that both expressions in the brackets in (13) are 0 and completes the proof. □

3 Geometric observations

We shall prove several facts.

Proposition 2. The feet of the perpendiculars from point Q, whose complex coordinate is given by (10), to the sides of $\triangle ABC$ form the smallest triangle of the set Φ.

Proof The similarity ratio between $\triangle X'Y'Z'$ and $\triangle XYZ$ is $\frac{|\mu|}{|t|}$, and it becomes smallest when $|\mu|$ is smallest, that is, when $\mu = -\frac{1}{2}$. Consider this minimal triangle for the set Φ. Using (14), we have

$$\begin{aligned} x' = \rho(x) &= -\frac{x-p}{2t} + q \\ &= -\frac{q + \bar{q}bc - b - c}{2} + q \\ &= \frac{q + b + c - \bar{q}bc}{2}, \end{aligned}$$

hence X' is the foot of the perpendicular from Q to BC. Similarly for Y' and Z'. □

Observe that Q, as the image of P under all affine maps ρ, is fixed for the whole set Φ, and that all triangles in Φ could be obtained from one another by direct similarity centered at Q. If $\mu = -\frac{1 + i\tan(\phi)}{2}$, with real part $-\frac{1}{2}$ and argument $180° + \phi$, the corresponding triangle under $\rho(w) = \frac{\mu}{t}(w-p)+q$ is obtained from the minimal one through direct similarity with angle ϕ and coefficient $\sec(\phi)$.

We shall prove the following statement about the fixed point Q for the family Φ.

Proposition 3. The point Q for Φ satisfies

$$\begin{aligned} \angle AQB &= \angle C + \angle Z, \\ \angle BQC &= \angle A + \angle X, \\ \angle CQA &= \angle B + \angle Y, \end{aligned}$$

where the angles above are directed. Furthermore,

$$AQ : BQ : CQ = \frac{YZ}{BC} : \frac{ZX}{CA} : \frac{XY}{AB}.$$

Proof Consider the ratio

$$s = \frac{b-q}{a-q} \times \frac{x-z}{y-z} \times \frac{a-c}{b-c}, \tag{15}$$

whose argument is $\angle AQB - \angle C - \angle Z$ (where the angles are directed).

From (10), we have

$$\begin{aligned} s &= \frac{byz(c-b) + bxy(b-a) - xyc(b-a) - yza(c-b)}{azx(a-c) + axy(b-a) - xyc(b-a) - zxb(a-c)} \times \frac{x-z}{y-z} \times \frac{a-c}{b-c} \\ &= \frac{yz(b-a)(c-b) + xy(b-c)(b-a)}{zx(a-b)(a-c) + xy(a-c)(b-a)} \times \frac{x-z}{y-z} \times \frac{a-c}{b-c} \\ &= \frac{y(z-x)(b-a)(c-b)}{x(z-y)(a-b)(a-c)} \times \frac{x-z}{y-z} \times \frac{a-c}{b-c} \\ &= \frac{y}{x}\frac{(x-z)^2}{(y-z)^2} = \frac{(x-z)(\bar{x}-\bar{z})}{(y-z)(\bar{y}-\bar{z})} = \frac{|x-z|^2}{|y-z|^2} > 0 \end{aligned}$$

Therefore, $s \in \mathbb{R}^+$ with $\arg(s) = 0$, hence the first of the angular identities holds (the other two follow similarly). At the same time, $|s| = \frac{BQ}{AQ} \cdot \frac{XZ}{YZ} \cdot \frac{AC}{BC}$, hence

$$\frac{BQ}{AQ} \cdot \frac{XZ}{YZ} \cdot \frac{AC}{BC} = \frac{XZ^2}{YZ^2},$$

That proves the second part of the proposition. □

The minimal triangle in the family is therefore the pedal triangle of Q. All other triangles in Φ are obtained from it by direct similarities centered at Q.

4 Special cases

We saw that if $\triangle XYZ$ is similar to the mirror image of $\triangle ABC$, the set Φ is empty. More interesting are the cases where $\triangle XYZ$ is similar to one of the triangles $\triangle ABC, \triangle BCA$, or $\triangle CAB$. Consider the sets $\Phi_1 = \Phi_{\triangle ABC}(\triangle BCA)$ and $\Phi_2 = \Phi_{\triangle ABC}(\triangle CAB)$. Using (9), (10), and (5), we can show that

$$t_1 = \frac{3abc - ab^2 - bc^2 - ca^2}{(a-b)(b-c)(c-a)},$$
$$t_2 = \frac{ac^2 + cb^2 + ba^2 - 3abc}{(a-b)(b-c)(c-a)},$$
$$p_1 = q_1 = \frac{a^2b^2 + b^2c^2 + c^2a^2 - abc(a+b+c)}{ac^2 + cb^2 + ba^2 - 3abc},$$
$$p_2 = q_2 = \frac{a^2b^2 + b^2c^2 + c^2a^2 - abc(a+b+c)}{ab^2 + bc^2 + ca^2 - 3abc}$$

Notice that

$$|t_1| = |t_2|,$$
$$|p_1| = |q_1| = |p_2| = |q_2|.$$

The equality $|t_1| = |t_2|$ implies that the feet of the perpendiculars from Q_1 and Q_2 onto the sides of $\triangle ABC$ form congruent triangles, as both are minimal for Φ_1 and Φ_2 and obtained from $\triangle ABC$ through direct similarities with magnitude $\frac{1}{2|t|}$. The fact that $|q_1| = |q_2|$ shows the fixed points for Φ_1 and Φ_2 are equidistant from the circumcenter of $\triangle ABC$.

Proposition 4. The points Q_1 and Q_2 are internal for $\triangle ABC$ and satisfy the following identities:

$$\angle Q_1AB = \angle Q_1BC = \angle Q_1CA = \theta,$$
$$\angle Q_2BA = \angle Q_2CB = \angle Q_2AC = \theta,$$
$$\angle Q_1OQ_2 = 2\theta$$

where $\theta \leq 30°$ is the solution of the equation

$$\tan(\theta) = \frac{\sin(\angle A)\sin(\angle B)\sin(\angle C)}{\cos(\angle A)\cos(\angle B)\cos(\angle C) + 1}.$$

Proof Consider Q_1 first. From Proposition 3, it follows that the directed angles $\angle AQ_1B, \angle BQ_1C$, and $\angle CQ_1A$ are $\angle C + \angle A$, $\angle A + \angle B$, and $\angle B + \angle C$ respectively. Those are less than $180°$, hence Q_1 is internal for $\triangle ABC$. Now, if $\theta = \angle Q_1AB$, we get $\angle Q_1BA = \angle B - \theta$ so $\angle Q_1BC = \theta$. Similarly, $\angle Q_1CA = \theta$.

Next, to determine θ, consider the argument of the following complex number, whose argument is $-\theta$:

$$\begin{aligned}\frac{b-a}{q_1-a} &= \frac{(ac^2+cb^2+ba^2-3abc)(b-a)}{a^2b^2+b^2c^2+c^2a^2-abc(a+b+c)-a(ac^2+cb^2+ba^2-3abc)} \\ &= \frac{(ac^2+cb^2+ba^2-3abc)(b-a)}{b(a^2+c^2-2ac)(b-a)} \\ &= \frac{(ac^2+cb^2+ba^2-3abc)}{b(a^2+c^2-2ac)} \\ &= \frac{\frac{c}{b}+\frac{b}{a}+\frac{a}{c}-3}{\frac{a}{c}+\frac{c}{a}-2} \\ &= \frac{3-\frac{c}{b}-\frac{b}{a}-\frac{a}{c}}{|a-c|^2} \\ &= \frac{(3-\cos(2\angle A)-\cos(2\angle B)-\cos(2\angle C))-i(\sin(2\angle A)+\sin(2\angle B)+\sin(2\angle C))}{|a-c|^2}\end{aligned}$$

Therefore,

$$\begin{aligned}\tan(-\theta) &= -\frac{\sin(2\angle A)+\sin(2\angle B)+\sin(2\angle C)}{3-\cos(2\angle A)-\cos(2\angle B)-\cos(2\angle C)} \\ &= -\frac{4\sin(\angle A)\sin(\angle B)\sin(\angle C)}{2\sin^2(\angle A)+2\sin^2(\angle B)+2\sin^2(\angle C)} \\ &= -\frac{\sin(\angle A)\sin(\angle B)\sin(\angle C)}{\cos(\angle A)\cos(\angle B)\cos(\angle C)+1}\end{aligned}$$

The proof for Q_2 is analogous and yields the same θ, since the solution of the equation is unique. Next, to prove that $\angle Q_1OQ_2 = 2\theta$, consider the ratio

$$\begin{aligned}\frac{q_2}{q_1} &= \frac{ac^2+cb^2+ba^2-3abc}{ab^2+bc^2+ca^2-3abc} \\ &= \frac{3-\frac{c}{b}-\frac{b}{a}-\frac{a}{c}}{3-\frac{b}{c}-\frac{c}{a}-\frac{a}{b}},\end{aligned}$$

whose argument as we saw above is $-\theta-\theta=-2\theta$.

To prove the last bit, namely that $\theta \leq 30°$, consider Sine Ceva's theorem for Q_1:

$$\sin^3(\theta) = \sin(\angle A-\theta)\sin(\angle B-\theta)\sin(\angle C-\theta).$$

Taking logarithms on both sides and observing that the function $\ln(\sin(x))$ is concave for $x \in (0, \pi)$, we have

$$\begin{aligned} 3\ln(\sin(\theta)) &= \ln(\sin(\angle A - \theta)) + \ln(\sin(\angle B - \theta)) + \ln(\sin(\angle C - \theta)) \\ &\leq 3\ln\left(\sin\left(\frac{\angle A - \theta + \angle B - \theta + \angle C - \theta)}{3}\right)\right) \\ &= 3\ln(\sin(60^\circ - \theta)) \end{aligned}$$

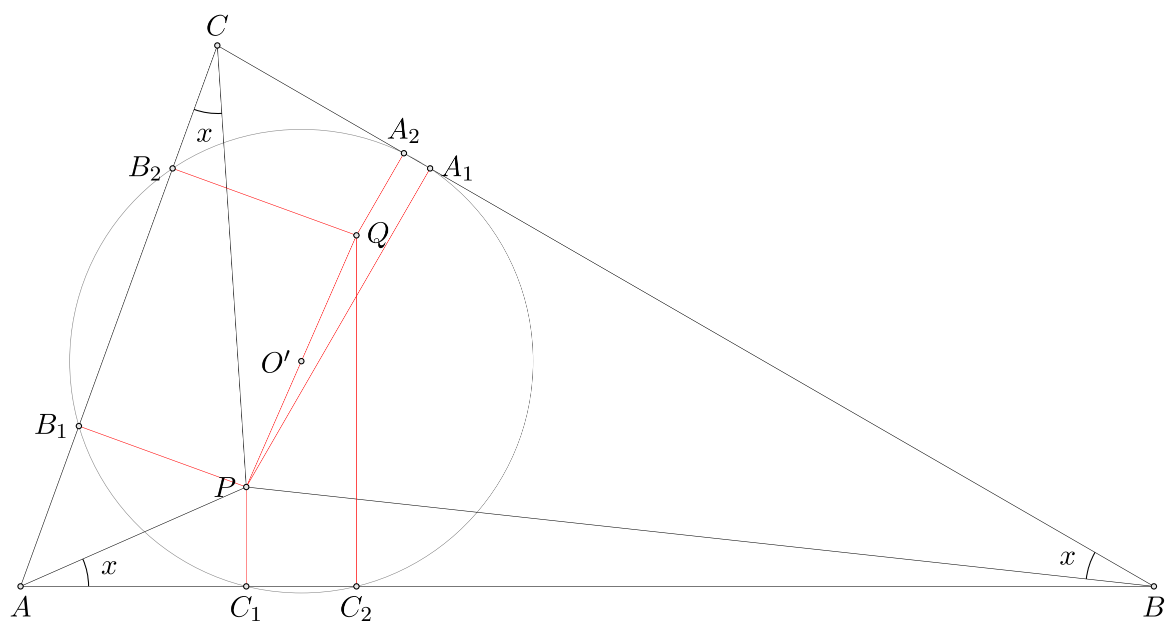
Notice that $\theta \leq \angle A, \angle B, \angle C$, hence $\theta \in (0, 60^\circ)$, where $\ln(\sin(x))$ is increasing, hence $\theta \leq 60^\circ - \theta$ and the proof is complete. □

All those results can be nicely formulated as a problem.

Problem 1. Points P and Q inside an acute-angled $\triangle ABC$ are such that $\angle PCA = \angle PAB = \angle PBC$ and $\angle QAC = \angle QBA = \angle QCB$. Prove that

(i) the feet of the perpendiculars from P and Q to the sides of $\triangle ABC$ lie on a circle whose center is the midpoint of PQ.

(ii) if O is the circumcenter of $\triangle ABC$, then $PO = QO$.



5 Outscribed triangles

Consider again $\triangle ABC$ and $\triangle XYZ$ inscribed in the unit circle, and let $\Sigma_{\triangle ABC}(\triangle XYZ) = \Sigma$ denote all triangles similar to $\triangle XYZ$ such that ABC is inscribed inside them. We shall find the general form of the affine maps sending $\triangle XYZ$ into triangles $\triangle X''Y''Z''$ from Σ.

Consider such a transform σ and let $\sigma(w) = uw+v$. We have $\sigma(\triangle XYZ) = \triangle X''Y''Z''$, hence $\sigma^{-1}(\triangle X''Y''Z'') = \triangle XYZ$. As $\triangle ABC$ is inscribed in $\triangle X''Y''Z''$, it follows that $\triangle A'B'C' = \sigma^{-1}(\triangle ABC)$ is inscribed in $\sigma^{-1}(\triangle X''Y''Z'') = \triangle XYZ$, hence $\triangle A'B'C' \in \Phi_{\triangle XYZ}(\triangle ABC)$.

Using the formula (11), we get

$$\sigma^{-1}(w) = \frac{1}{u}w - \frac{v}{u} = \frac{\mu}{t^*}(w - p^*) + q^*,$$

where p^*, q^*, t^* are obtained from the same formulas as p, q, t in (5),(9),(10) but with the triples x, y, z and a, b, c interchanged. Thus,

$$\sigma(w) = uw + v = \frac{t^*}{\mu}(w - q^*) + p^*. \tag{16}$$

We shall examine the image of any given point $W \neq Q^*$ under the various transforms σ. Let its complex coordinate be w, and let $w' = \sigma(w)$. From (16), we have

$$w' = p^* + \frac{t^*(w - q^*)}{\mu}.$$

Set

$$\alpha = t^*(w - q^*), \qquad \xi = \frac{1}{\mu}.$$

Then

$$w' = p^* + \alpha\xi.$$

Now $\Re(\mu) = -\frac{1}{2}$, so if $\xi = u + iv$, then $\Re\left(\frac{1}{\xi}\right) = \Re(\mu) = -\frac{1}{2}$. Since

$$\frac{1}{\xi} = \frac{\bar{\xi}}{|\xi|^2} = \frac{u - iv}{u^2 + v^2},$$

it follows that

$$\frac{u}{u^2 + v^2} = -\frac{1}{2} \Rightarrow u^2 + v^2 + 2u = 0 \Leftrightarrow (u+1)^2 + v^2 = 1.$$

Therefore ξ runs over the circle with diameter endpoints 0 and -2. Multiplying by α and translating by p^*, we conclude that w' runs over the circle with diameter endpoints

$$p^* \qquad \text{and} \qquad p^* - 2t^*(w - q^*).$$

If $w = q^*$, then $\alpha = 0$ and the image degenerates to the single point p^*.

We conclude that the whole family of transforms has the common fixed point P^*, and for every fixed point $W \neq Q^*$ its image is a circle passing through P^*.

6 Hyperbola

Consider the special case $\triangle XYZ \sim \triangle ABC$ and the families Φ and Σ of inscribed and outscribed triangles to $\triangle ABC$ similar to it.

Notice that in that case $t = t^* = 1$, $p = p^* = h$ (where H is the orthocenter of $\triangle ABC$), and $q = q^* = 0$.

For every $\phi \in (-90°, 90°)$, consider

$$\mu_1 = -\frac{1 + i\tan(\phi)}{2}, \mu_2 = -\frac{1 - i\tan(\phi)}{2},$$

The corresponding transforms ρ and σ as defined in (11) and (16) become

$$\rho(w) = \mu_1(w - h),$$
$$\sigma(w) = \frac{1}{\mu_2}w + h$$

The corresponding triangles from Φ and Σ under (11) and (16) are obtained from the same triangle by direct similarities with coefficients μ_1 and $\frac{1}{\mu_2}$ respectively. The ratio of those coefficients, $\mu_1\mu_2$, is real for every ϕ, hence the two triangles are homothetic. We shall examine this homothety as ϕ varies between $-90°$ and $90°$. Suppose its center is s and its coefficient is λ, both functions of ϕ. We have

$$\frac{\rho(a) - s}{\sigma(a) - s} = \lambda = \mu_1\mu_2,$$

Plugging in the values for $\rho(a)$ and $\sigma(a)$, we solve the linear equation for s to get

$$s = h\frac{\mu_1 + \mu_1\mu_2}{\mu_1\mu_2 - 1},$$

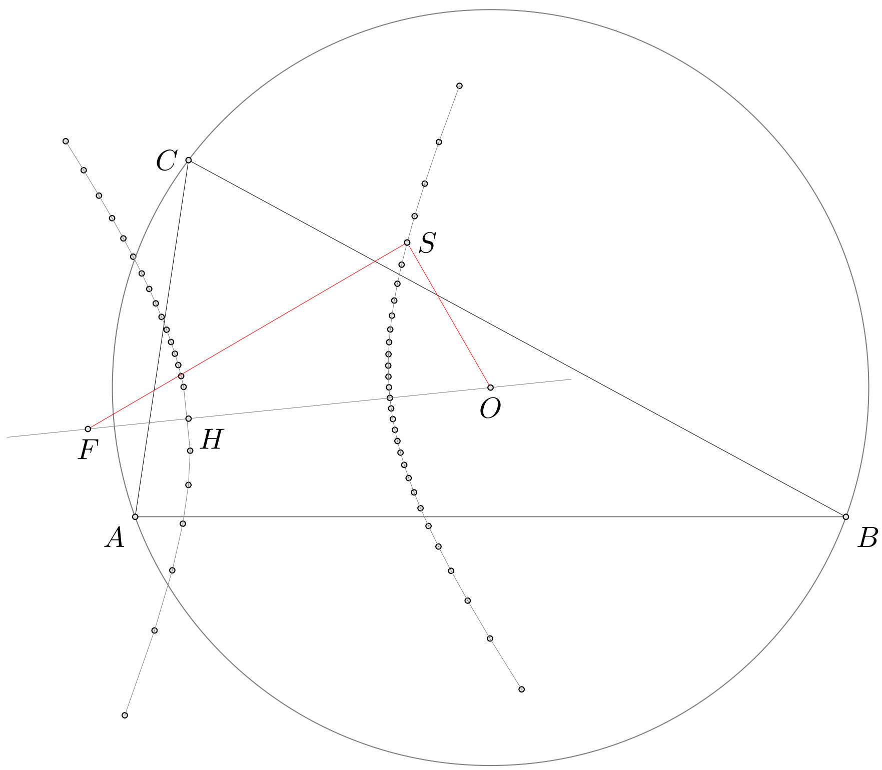
hence, if we set $\tan(\phi) = r$,

$$\frac{s}{h} = \frac{1-r^2}{3-r^2} + i\frac{2r}{3-r^2}.$$

We can show that

$$\left|s - \frac{4}{3}h\right| - |s| = \pm\left|\frac{2}{3}h\right|,$$

where we take sign $+$ when $r^2 < 3$ and sign $-$ when $r^2 > 3$. The case $r^2 = 3$ corresponds to the case when the inscribed and outscribed triangles are the same, hence the homothety is a translation and its center is a point at infinity. In all other cases, s lies on a hyperbola with foci O and the point symmetric to O across the midpoint of GH, both lying on the Euler line. Which leg of the hyperbola s lies on is determined by whether $r^2 < 3$ or $r^2 > 3$, equivalently whether $|\phi| < 60°$ or $|\phi| > 60°$.



This last result can be stated as a problem.

Problem 2. Let $\triangle ABC$ be a non-equilateral triangle with circumcenter O and orthocenter H. Let F lie on ray $\overrightarrow{OH}$ such that $OF : OH = 4 : 3$.

Points A_1, B_1, C_1 lie on BC, CA, AB respectively such that $\triangle A_1B_1C_1 \sim \triangle ABC$. The lines through A, B, C parallel to B_1C_1, C_1A_1, A_1B_1 intersect pairwise at C_2, A_2, B_2, respectively.

Prove that the lines A_1A_2, B_1B_2, and C_1C_2 concur at a point S satisfying

$$SF - SO = \frac{2}{3}OH.$$