Various problems

Vladimir Barzov

Abstract

These are problems that are not suitable for competitions for one reason or another, yet they are interesting in their own right.

Problem statements

Problem 1. Let

$$x_n = \underbrace{\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}_{n \text{ square roots}},$$

for $n \geq 1$. Prove that

$$\lim_{n\to\infty} 4^{n+1}(2-x_n) = \pi^2.$$

Problem 2. Does there exist a coloring of the vertices of a regular polygon in more than one color such that:

1. each color class is the vertex set of a regular polygon; and
2. the union of any two or more congruent monochromatic polygons is *not* the vertex set of a regular polygon?

For clarity, a pair of opposite vertices is regarded as a regular 2-gon, whereas a single vertex is not permitted as a regular 1-gon.

Problem 3. Given $2n$ distinct numbers, we want to partition them into n pairs so that the largest pair sum is as small as possible. Prove that the following pairing provides an optimal solution (though not necessarily the unique one): pair the smallest with the largest, the second smallest with the second largest, etc.

Problem 4. Let $n \geq 3$, and let $A_1A_2 \cdots A_n$ be a regular polygon of side length 1. Prove that, for any subset of

$$\left\{\overrightarrow{A_1A_2}, \overrightarrow{A_2A_3}, \ldots, \overrightarrow{A_{n-1}A_n}\right\},$$

the length of the sum of its vectors is less than

$$\frac{\pi}{n} + \frac{n}{\pi}.$$

Solutions

Solution 1. For convenience, define $x_0 = 0$. Then

$$x_{n+1} = \sqrt{2 + x_n}$$

for $n \geq 0$. Define

$$\alpha_n = \frac{\pi}{2^{n+2}}.$$

We claim that

$$x_n = 2 - 4\sin^2 \alpha_n$$

for all $n \geq 0$. For $n = 0$,

$$x_0 = 0 = 2 - 4\sin^2 \frac{\pi}{4} = 2 - 4\sin^2 \alpha_0.$$

Now assume the formula holds for some n. Then

$$\begin{aligned} x_{n+1} &= \sqrt{2 + x_n} = \sqrt{2 + 2 - 4\sin^2(\alpha_n)} = 2\cos \alpha_n \\ &= 2\cos(2\alpha_{n+1}) = 2\left(\cos^2 \alpha_{n+1} - \sin^2 \alpha_{n+1}\right) \\ &= 2 - 4\sin^2 \alpha_{n+1}, \end{aligned}$$

which completes the induction. Therefore,

$$\begin{aligned} 4^{n+1}(2 - x_n) &= 2^{2n+2} 4\sin^2(\alpha_n) = \frac{\pi^2}{\alpha_n^2} \sin^2 \alpha_n \\ &= \pi^2 \left(\frac{\sin \alpha_n}{\alpha_n}\right)^2 \to \pi^2 \end{aligned}$$

as $n \to \infty$ and $\alpha_n \to 0$. □

Solution 2. Yes, such a coloring exists. Let $n = 60$ and consider a regular n-gon with vertices labeled $0, 1, \ldots, n-1$. The vertices of any regular d-gon among the vertices of this regular n-gon can be described as

$$\{a + (n/d)i \mid 0 \leq i \leq d - 1\}$$

for some $0 \leq a \leq n/d - 1$. A necessary condition is $d \mid n$.

Now let us color the n vertices so that the color classes are exactly the following sets.

- Decagons: $P_j = \{a_j + 6i \mid 0 \leq i \leq 9\}$ for $a_j = 0, 2$.

- Hexagons: $P_j = \{a_j + 10i \mid 0 \leq i \leq 5\}$ for $a_j = 1, 3, 5, 7$.
- Squares: $P_j = \{a_j + 15i \mid 0 \leq i \leq 3\}$ for $a_j = 4$.
- 2-gons: $P_j = \{a_j + 30i \mid i = 0, 1\}$ for $a_j = 9, 10, 16, 22, 28, 29$.

It is tedious but straightforward to verify that these sets form a partition of the n vertices, so the first condition is satisfied. What remains is to show that the second condition is satisfied.

Observe the following: if the union of m d-gons described as

$$P_j = \{a_j + (n/d)i \mid 0 \leq i \leq d-1\}, j = 0, 1, \ldots, m-1$$

is a regular polygon, it must be a md-gon described as

$$\{a + (n/md)i \mid 0 \leq i \leq md - 1\}$$

for some $0 \leq a \leq n/md - 1$. Since both $\{a_0, a_1, \ldots, a_{m-1}\}$ and $\{a + (n/md)i \mid 0 \leq i \leq m-1\}$ are sets of m distinct integers between 0 and $n/d - 1$, we must have

$$\{a_0, a_1, \ldots, a_{m-1}\} = \{a + (n/md)i \mid 0 \leq i \leq m-1\}.$$

In other words, the set $\{a_0, a_1, \ldots, a_{m-1}\}$ must be an arithmetic progression with common difference n/md.

Now, consider each congruence class with more than one member:

First, consider the two decagons with $d = 10$. Their union has $m = 2$, and $\{a_0, a_1\} = \{0, 2\}$ which is not an arithmetic progression with common difference $n/md = 3$, hence it is not a regular polygon.

Next, consider the four hexagons with $d = 6$. If a union of m of them were a regular md-gon, we would have $m \mid n/d = 10$ and $2 \leq m \leq 4$, hence $m = 2$. Furthermore, the a_j would need to be an arithmetic progression with common difference $n/md = 5$, but no two elements of $\{1, 3, 5, 7\}$ have a difference of 5. Therefore, the union could not be a regular polygon.

Finally, consider the six 2-gons with $d = 2$. If a union of m of them were a regular md-gon, we would need $m \mid n/d = 30$ and $2 \leq m \leq 6$, so $m \in \{2, 3, 5, 6\}$. Examine each possibility.

- For $m = 2$, we would need two a_j among $9, 10, 16, 22, 28, 29$ with common difference $n/md = 15$, but no such exist.
- For $m = 3$, we would need three a_j with common difference $n/md = 10$, but no such exist.

- For $m = 5$, we would need five a_j with common difference $n/md = 6$, but no such set exists (though there is a set of 4 such a_j).
- For $m = 6$, we would need all six a_j to be an arithmetic progression with common difference $n/md = 5$, which is not the case.

That shows that the union of any two or more monochromatic polygons from the same congruence class is not a regular polygon, hence condition 2 is also satisfied. $\square$

Solution 3. Let the $2n$ numbers be

$$a_1 < a_2 < \cdots < a_{2n}.$$

For every partition G into pairs

$$(a_{i_1}, a_{j_1}), (a_{i_2}, a_{j_2}), \ldots, (a_{i_n}, a_{j_n}),$$

define

$$f(G) = \max_{1 \le k \le n} (a_{i_k} + a_{j_k}), \qquad s(G) = \sum_{k=1}^{n} i_k j_k.$$

Let G^* be the pairing described in the statement, i.e.

$$G^* = (a_1, a_{2n}), (a_2, a_{2n-1}), \ldots, (a_n, a_{n+1}).$$

We shall show that, starting from any pairing $G_0 \neq G^*$, we can successively swap two numbers from two pairs at each step, so that the resulting sequence of pairings $G_1, G_2, \ldots, G_m = G^*$ satisfies:

$$f(G_0) \ge f(G_1) \ge \cdots \ge f(G_m) = f(G^*),$$

$$s(G_0) > s(G_1) > \cdots > s(G_m).$$

This automatically proves the optimality of $f(G^*)$.

Let k be the minimal positive integer for which a_k is not paired with a_{2n+1-k}. Such k necessarily exists, since $G_0 \neq G^*$. For brevity, let $u = 2n + 1 - k$. Given that $a_1, a_2, \ldots, a_{k-1}$ are already paired with $a_{2n}, a_{2n-1}, \ldots, a_{u+1}$, and that a_k is not paired with a_u, two things follow:

- a_k is paired with some a_j where $k < j < u$;
- a_u is paired with some a_i where $k < i < u$.

Let us take the pairs (a_k, a_j) and (a_i, a_u) and swap a_j with a_u, obtaining the pairs (a_k, a_u) and (a_i, a_j). This new pairing G_1 differs from G_0 in only those two pairs, so we only need to examine how f and s change for these two pairs. From $a_i > a_k$ and $a_u > a_j$ we have

$$a_i + a_u > \begin{cases} a_k + a_u \\ a_i + a_j \end{cases},$$

so

$$\max(a_k + a_j, a_i + a_u) = a_i + a_u > \max(a_k + a_u, a_i + a_j),$$

hence $f(G_0) \geq f(G_1)$, since outside those two pairs, nothing changes. At the same time,

$$s(G_0) - s(G_1) = (kj + iu) - (ku + ij) = (u - j)(i - k) > 0.$$

Similarly, if $G_1 \neq G^*$, we repeat the procedure and get G_2 with $f(G_1) \geq f(G_2)$ and $s(G_1) > s(G_2)$ and so on. But $s(G_0), s(G_1), \ldots$ is a strictly decreasing sequence of positive integers, so the process terminates after finitely many steps. But the only way for it to terminate is if $G_m = G^*$. This completes the proof. $\square$

Solution 4. Let $k = \lfloor n/2 \rfloor$, and let γ, R, O be the circumcircle, circumradius, and circumcenter of the polygon. If $\alpha = \pi/n$, from $\triangle OA_1A_2$ with $\angle A_1OA_2 = 2\alpha$, we have

$$R = \frac{1}{2\sin\alpha}.$$

Let

$$A = \{a_i = \overrightarrow{A_iA_{i+1}} \mid i = 1, 2, \ldots, n\},$$

where the index i runs cyclically on n. For any set of vectors, we call the length of its sum the sum-length. The objective is to determine the maximum sum-length among subsets of A. Though the problem statement explicitly excludes a_n, we can include it for symmetry without changing the result. Indeed, the sum-length of A is 0 and for any $A' \subseteq A$ with $a_n \in A'$ its complement $A'' = A \setminus A'$ has the same sum-length.

First, we shall show that

$$A_1A_{k+1} < \frac{\pi}{n} + \frac{n}{\pi},$$

and then we shall prove that the set

$$\{a_1, a_2, \ldots, a_k\},$$

whose sum-length is A_1A_{k+1}, has the maximal sum-length. These two claims together complete the proof.

To prove the upper bound for A_1A_{k+1}, consider $\triangle A_1OA_{k+1}$, where

$$A_1A_{k+1} = 2R\sin(\angle A_1OA_{k+1}/2) = 2R\cos(\angle A_1A_{k+1}O).$$

We have

$$\angle A_1A_{k+1}O = \begin{cases} 0 & \text{for even } n \\ \alpha/2 & \text{for odd } n \end{cases}.$$

Therefore, for even n:

$$A_1A_{k+1} = 2R = \frac{1}{\sin\alpha},$$

and for odd n:

$$A_1A_{k+1} = 2R\cos(\alpha/2) = \frac{\cos(\alpha/2)}{\sin\alpha}.$$

Since $\cos(\alpha/2) \leq 1$, in either case,

$$A_1A_{k+1} \leq \frac{1}{\sin\alpha}.$$

Notice that A_1A_{k+1} has maximal possible length among the chords determined by the vertices of the polygon, as $\angle A_1OA_{k+1}$ is maximal among the central angles subtending such chords.

Now, using the Taylor expansion for $\sin\alpha$, and since $\alpha = \pi/n \in (0, 2)$, we have

$$\sin\alpha = \alpha - \frac{\alpha^3}{3!} + \sum_{i\geq 3}\left(\frac{\alpha^{2i-1}}{(2i-1)!} - \frac{\alpha^{2i+1}}{(2i+1)!}\right) > \alpha - \frac{\alpha^3}{6},$$

since for $\alpha \in (0, 2)$ each bracketed summand is positive. Therefore, using $\alpha^2 < 5$,

$$A_1A_{k+1} \leq \frac{1}{\sin\alpha} < \frac{1}{\alpha - \alpha^3/6} < \alpha + \frac{1}{\alpha} = \frac{\pi}{n} + \frac{n}{\pi},$$

which is what we wanted.

What remains is to prove that A_1A_{k+1} is the maximum obtainable sum-length. Assume the contrary, and let $B \subseteq A$ be a subset with maximal sum-length. If $b = \sum_{a\in B} a$, then

$$|b| > A_1A_{k+1}.$$

Let ϕ_i be the angle between a_i and b for $i = 1, 2, \ldots, n$ and

$$\Phi = (\phi_i \mid 1 \leq i \leq n) = (\xi + 2i\alpha \mid 1 \leq i \leq n)$$

where $\xi = \phi_n$, and each $\xi + 2i\alpha$ is taken modulo 2π.

Now, observe that for every $a_i \in B$ we must have $|b| \geq |b - a_i|$, else we could remove a_i from B and get an even longer sum-length. Thus

$$|b|^2 \geq |b - a_i|^2 = |a_i|^2 - 2a_i \cdot b + |b|^2 = 1 - 2|b| \cos \phi_i + |b|^2$$

$$\Rightarrow 2|b| \cos \phi_i \geq 1 \Rightarrow \cos \phi_i \geq \frac{1}{2|b|} > 0$$

$$\Rightarrow \phi_i \notin \Delta_1 = \left[\frac{\pi}{2}, \frac{3\pi}{2}\right].$$

Therefore, for any $1 \leq i \leq n$, we must have

$$\phi_i \in \Delta_1 \Rightarrow a_i \notin B. \qquad (*)$$

Likewise, for any $a_i \in A \setminus B$ we must have $|b| \geq |b + a_i|$, else we could add a_i to B and get a strictly larger sum-length. Thus

$$|b|^2 \geq |b + a_i|^2 = |a_i|^2 + 2a_i \cdot b + |b|^2 = 1 + 2|b| \cos \phi_i + |b|^2$$

$$\Rightarrow -1 \geq 2|b| \cos \phi_i \Rightarrow \cos \phi_i \leq \frac{-1}{2|b|} < 0$$

$$\Rightarrow \phi_i \notin \Delta_2 = \left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right).$$

Therefore, for any $1 \leq i \leq n$, we must have

$$\phi_i \in \Delta_2 \Rightarrow a_i \in B. \qquad (**)$$

When viewed modulo 2π, both Δ_1 and Δ_2 are connected arcs on the circle. Since $\Delta_1 \cup \Delta_2 = [0, 2\pi)$, every element $\phi_i \in \Phi$ must fall in either Δ_1 or Δ_2. No element ϕ_i can belong to both, otherwise its corresponding a_i would have to both belong to B and not belong to B, which is impossible. Since the ϕ_i are cyclically ordered, the elements of Φ lying in each of the arcs Δ_1 and Δ_2 form a cyclically consecutive block. Thus:

- the elements $\phi_i \in \Delta_1$ form a cyclically consecutive block, and by $(*)$ their corresponding vectors satisfy $a_i \notin B$;
- the elements $\phi_i \in \Delta_2$ form a cyclically consecutive block, and by $(**)$ their corresponding vectors satisfy $a_i \in B$.

Since every element of Φ lies in exactly one of Δ_1 and Δ_2, these two blocks together exhaust all elements of Φ. Hence the elements of B form one cyclically consecutive block, and therefore b is a chord, i.e.

$$b = \overrightarrow{A_i A_j}$$

for some i and j. But as $A_1 A_{k+1}$ is the longest of all chords, we get $|b| \leq A_1 A_{k+1}$, a contradiction with the assumption that $A_1 A_{k+1}$ is not the maximal sum-length. $\square$